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Some log-majorizations and an extension of a determinantal inequality

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ABSTRACT

An eigenvalue inequality involving a matrix connection and its dual is established, and some log-majorization type results are obtained. In particular, some eigenvalues inequalities considered by F. Hiai and M. Lin [9], an associated conjecture, and a singular values inequality by L. Zou [20] are revisited. A reformulation of the inequality $\det(A+U^*B) \leq \det(A+B)$, for positive semidefinite matrices A, B, with U a unitary matrix that appears in the polar decomposition of $B\,A$, is also extended, using some known norm inequalities, associated to Furuta inequality and Araki–Cordes inequality.

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1. Introduction

Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be vectors with the components sorted in non-increasing order, that is, $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$. We say that $\mathbf{y}$ *weakly majorizes* $\mathbf{x}$ and write $\mathbf{x} \prec_{\mathrm{w}} \mathbf{y}$ if

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \qquad k = 1, \ldots, n. \tag{1}$$

If $\mathbf{x} \prec_{\mathrm{w}} \mathbf{y}$ and equality holds in (1) for $k = n$, we say that $\mathbf{y}$ *majorizes* $\mathbf{x}$, denoted by $\mathbf{x} \prec \mathbf{y}$. For $\mathbf{x}, \mathbf{y}$ with nonnegative components, we write $\mathbf{x} \prec_{\log} \mathbf{y}$ if $\mathbf{y}$ log-*majorizes* $\mathbf{x}$, that is,

$$\prod_{i=1}^{k} x_i \leq \prod_{i=1}^{k} y_i, \qquad k = 1, \ldots, n, \tag{2}$$

with equality occurring in (2) when $k = n$.

For any real valued function f defined on an interval, containing all the components of the real vector $\mathbf{x}$, we adopt the notation $f(\mathbf{x}) = (f(x_1), ..., f(x_n))$. If all the components of $\mathbf{x}, \mathbf{y}$ are positive, then $\mathbf{x} \prec_{\log} \mathbf{y}$ if and only if $\log \mathbf{x} \prec \log \mathbf{y}$, this justifying the log-majorization terminology. If f is convex, then $\mathbf{x} \prec \mathbf{y}$ implies $f(\mathbf{x}) \prec_{\mathrm{w}} f(\mathbf{y})$. In particular, the log-majorization implies the weak majorization. Additionally, if f is an increasing and convex function, then $\mathbf{x} \prec_{\mathrm{w}} \mathbf{y}$ implies $f(\mathbf{x}) \prec_{\mathrm{w}} f(\mathbf{y})$. For instance, $f(t) = \ln(1 + e^t)$ is a strictly increasing and convex function on $(0, +\infty)$. Two important resources on the topic of majorization are [2,15].

Let M_n be the algebra of $n \times n$ complex matrices and I be the identity matrix of order n. For $A \in M_n$ with real eigenvalues, we denote by $\lambda(A)$ the n-tuple of eigenvalues of A arranged as follows $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. If $A, B \in M_n$, then AB and BA have the same eigenvalues, including multiplicities [11, Theorem 1.3.20], hence $\lambda(AB) = \lambda(BA)$.

For simplicity of notation, if $A, B \in M_n$ have real eigenvalues, then we write $A \prec_{\mathrm{w}} B$ whenever $\lambda(A) \prec_{\mathrm{w}} \lambda(B)$; moreover, if $A, B \in M_n$ have nonnegative eigenvalues, we write $A \prec_{\log} B$ when $\lambda(A) \prec_{\log} \lambda(B)$. Majorization is a powerful tool for establishing determinantal and matrix norm inequalities. In particular, if $A \prec_{\log} B$, then $\det(I+A) \leq \det(I + B)$. On the other hand, some classical determinantal inequalities can find their majorization counterparts.

For $A \in M_n$, the unique positive semidefinite square root of A^*A is denoted by $|A|$. For $A, B \in M_n$, Ky Fan Dominance Theorem [15] asserts that $|A| \prec_{\mathrm{w}} |B|$ if and only if $|||A||| \leq |||B|||$ holds for any unitarily invariant norm $|||\cdot|||$ in M_n. We recall that a norm $|||\cdot|||$ is said to be *unitarily invariant* in M_n if $|||UAV||| = |||A|||$ for all $A \in M_n$ and all unitary matrices $U, V \in M_n$. Considering the singular values of $A \in M_n$, that is, the eigenvalues of $|A|$, ordered as follows $s_1(A) \geq \cdots \geq s_n(A)$, the Ky Fan k-norms of A defined by