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A note on factoring unitary matrices

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The note concerns three factorizations of unitary matrices: a recent factorization allowing to obtain any unitary matrix of dimension $2n$ from four unitary matrices of dimension n; the Cosine–Sine decomposition; and a new factorization decomposing a given unitary matrix into a product of two orthogonal matrices and a diagonal unitary matrix.

1. Introduction

The problem of factoring unitary matrices was treated by many researchers (see, e.g., [3,4,6,8–10,12,13,15,16,18,20,21]). In [5] a recent factorization of unitary matrices of an even dimension has been provided. Namely, that for every matrix U in the unitary group $U(2n)$ there exist $A, B, C, D \in U(n)$, such that

$$U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \frac{1}{2} \begin{bmatrix} I + C & I - C \\ I - C & I + C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}, \tag{1}$$

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where I is the identity matrix in $U(n)$. This factorization has been applied in [2] to synthesize an arbitrary quantum circuit. As indicated in [2] such synthesis requires an analysis part, that for a given $U \in U(2n)$ allows to find $A, B, C, D \in U(n)$, so that (1) holds. The authors of [2] have resolved this problem in a regular case, that is, for $U \in U(2n)$ written in the block form as

$$U = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix}$$

with $n \times n$ matrices X, Y, W, Z that are invertible.

In this note, the general case of finding the factorization (1) is treated (see Theorem 2.1) leading to a deeper understanding of the well known Cosine–Sine decomposition (see Corollary 3.1 and Corollary 3.2).

Also, factorization (1) is extended to the odd case $U \in U(2n+1)$ (see Theorem 4.1) and another factorization of unitary matrices is provided, that can be applied to synthesize an arbitrary unitary matrix (see Theorem 5.1).

2. Factorization of $U\,(2n)$

One has the following

Theorem 2.1. *Let $U \in U(2n)$ be a unitary matrix written in the block form as*

$$U = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} \in U(2n) \tag{2}$$

with the $n \times n$ matrices X, Y, W, Z. Then, for every polar decomposition $X = S_X U_X$ and $Y = S_Y U_Y$, one has that

$$U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \frac{1}{2} \begin{bmatrix} I + C & I - C \\ I - C & I + C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}, \quad A, B, C, D \in U(n), \tag{3}$$

with I-identity in $U(n)$ and A, B, C, D given by

$$\begin{aligned} A &:= (S_X - iJS_Y)U_X, & B &:= W - iZU_Y^* JU_X, \\ C &:= U_X^*(S_X + iJS_Y)^2 U_X, & D &:= iU_X^* JU_Y, \end{aligned} \tag{4}$$

where J is an arbitrary unitary matrix commuting with S_X and S_Y, such that $J^2 = I$.

Proof. In the polar decompositions $X = S_X U_X$ and $Y = S_Y U_Y$ the matrices S_X, S_Y are positive semi-definite (thus Hermitian) and U_X, U_Y are unitary. Since $XX^* + YY^* = I$ one gets that $S_X^2 + S_Y^2 = I$. Therefore, S_X and S_Y commute (since positive semi-definite), and