

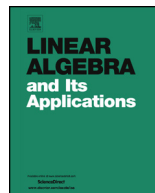


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## A note on factoring unitary matrices

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## ABSTRACT

The note concerns three factorizations of unitary matrices: a recent factorization allowing to obtain any unitary matrix of dimension  $2n$  from four unitary matrices of dimension  $n$ ; the Cosine–Sine decomposition; and a new factorization decomposing a given unitary matrix into a product of two orthogonal matrices and a diagonal unitary matrix.

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## 1. Introduction

The problem of factoring unitary matrices was treated by many researchers (see, e.g., [3,4,6,8–10,12,13,15,16,18,20,21]). In [5] a recent factorization of unitary matrices of an even dimension has been provided. Namely, that for every matrix  $U$  in the unitary group  $U(2n)$  there exist  $A, B, C, D \in U(n)$ , such that

$$U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \frac{1}{2} \begin{bmatrix} I + C & I - C \\ I - C & I + C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}, \quad (1)$$

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where  $I$  is the identity matrix in  $U(n)$ . This factorization has been applied in [2] to synthesize an arbitrary quantum circuit. As indicated in [2] such synthesis requires an analysis part, that for a given  $U \in U(2n)$  allows to find  $A, B, C, D \in U(n)$ , so that (1) holds. The authors of [2] have resolved this problem in a regular case, that is, for  $U \in U(2n)$  written in the block form as

$$U = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix}$$

with  $n \times n$  matrices  $X, Y, W, Z$  that are invertible.

In this note, the general case of finding the factorization (1) is treated (see Theorem 2.1) leading to a deeper understanding of the well known Cosine–Sine decomposition (see Corollary 3.1 and Corollary 3.2).

Also, factorization (1) is extended to the odd case  $U \in U(2n+1)$  (see Theorem 4.1) and another factorization of unitary matrices is provided, that can be applied to synthesize an arbitrary unitary matrix (see Theorem 5.1).

## 2. Factorization of $U(2n)$

One has the following

**Theorem 2.1.** *Let  $U \in U(2n)$  be a unitary matrix written in the block form as*

$$U = \begin{bmatrix} X & Y \\ W & Z \end{bmatrix} \in U(2n) \tag{2}$$

with the  $n \times n$  matrices  $X, Y, W, Z$ . Then, for every polar decomposition  $X = S_X U_X$  and  $Y = S_Y U_Y$ , one has that

$$U = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \frac{1}{2} \begin{bmatrix} I + C & I - C \\ I - C & I + C \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}, \quad A, B, C, D \in U(n), \tag{3}$$

with  $I$ -identity in  $U(n)$  and  $A, B, C, D$  given by

$$\begin{aligned} A &:= (S_X - iJS_Y)U_X, & B &:= W - iZU_Y^*JU_X, \\ C &:= U_X^*(S_X + iJS_Y)^2U_X, & D &:= iU_X^*JU_Y, \end{aligned} \tag{4}$$

where  $J$  is an arbitrary unitary matrix commuting with  $S_X$  and  $S_Y$ , such that  $J^2 = I$ .

**Proof.** In the polar decompositions  $X = S_X U_X$  and  $Y = S_Y U_Y$  the matrices  $S_X, S_Y$  are positive semi-definite (thus Hermitian) and  $U_X, U_Y$  are unitary. Since  $XX^* + YY^* = I$  one gets that  $S_X^2 + S_Y^2 = I$ . Therefore,  $S_X$  and  $S_Y$  commute (since positive semi-definite), and

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