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On the extreme eigenvalues of certain matrices of non-standard inner products of Hermite polynomials

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ABSTRACT

We study Hermite orthogonal polynomials and Gram matrices of their non-standard inner products. The weight function of the non-standard inner product is obtained from the Gauss probability density function by its horizontal shift by a real parameter. We are interested in the spectral properties of these matrices and some of their modifications. We show how the largest and smallest eigenvalues of the matrices depend on the parameter.

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1. Introduction

Orthogonal polynomials are widely used as bases of spaces for approximation functions in many theoretical and numerical disciplines; see for example [1,5,8,14]. In this paper we are interested in the Hermite orthogonal polynomials h_m, $m = 0, 1, \ldots$, which fulfill

$$\int_{\mathbb{R}} h_m(x) h_k(x) \mathrm{e}^{-\frac{x^2}{2}} \, \mathrm{d}x = 0 \quad \text{for } k \neq m,$$

where the subscript denotes the degree of the polynomial. We also consider a non-standard inner product on the set of all polynomials on $\mathbb{R}$. For a given constant $\alpha \in \mathbb{R}$ the non-standard inner product of two polynomials f and g is defined as

$$\langle f, g \rangle_\alpha = \int_{\mathbb{R}} f(x) g(x) \mathrm{e}^{-\frac{x^2}{2} + \alpha x} \, \mathrm{d}x.$$

We take the $N + 1$ Hermite polynomials $h_0, \ldots, h_N$ and build the $(N + 1) \times (N + 1)$ Gram matrix A_α with elements $(A_\alpha)_{m+1,k+1} = \langle h_m, h_k \rangle_\alpha$. After diagonal scaling (see, e.g. [7,16]) of A_α we obtain $\overline{A}_\alpha$. The goal of this paper is to study the spectral properties of $\overline{A}_\alpha$. Among other results, we show that for $\alpha \in [0, \infty)$ or $\alpha \in (-\infty, 0]$, respectively, the extreme eigenvalues of $\overline{A}_\alpha$ or some of their bounds depend monotonically on α.

The motivation for this research comes from the numerical solution of problems with uncertain (random) data; in particular, from their preconditioning and a posteriori estimates of errors. However, the presented study can be applied irrespective of our motivation, therefore, readers interested only in the results (and not in the motivation) may skip the next paragraph.

One popular method for the numerical solution of elliptic partial differential equations with uncertain data is the stochastic Galerkin method, which leads to the solution of very large and possibly ill-conditioned systems of linear equations; see e.g. [2,18]. Structure and spectral properties of the matrices of the systems depend on the functions which are used for the approximation of the solution. While for the physical part of the solution the standard finite element functions are usually employed, polynomials are appropriate to approximate the stochastic part of the solution. The approximation space V is then the tensor product of the space of finite element functions and of polynomials of random variables. The bases of the polynomials are chosen according to the probability distribution of the random data. In many physical applications, the uncertain coefficients of differential equations are considered as log-normal random fields; see [10], and [2,6,15] for some theoretical conclusions. For the purpose of numerical computation, the logarithm of the coefficient is expressed in the form of the truncated Karhunen–Lòeve expansion $a_0(x) + \sum_{i=1}^{M} a_i(x) y_i$, where the a_i are functions (of the physical variable x) given by the random field, and the y_i are independent and normally distributed random variables [2,13]. (In the description of our motivation, we skip many details; for example, the origin of the functions a_i, which is not important for the study presented in this paper.)