Linear Algebra and its Applications 546 (2018) 154–168

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The critical group of the Kneser graph on 2-subsets of an n-element set ☆

Joshua E. Ducey [a,*], Ian Hill [a], Peter Sin [b]

[a] Dept. of Mathematics and Statistics, James Madison University, Harrisonburg, VA 22807, USA
[b] Dept. of Mathematics, University of Florida, P.O. Box 118105, Gainesville, FL 32611, USA

A R T I C L E   I N F O

Article history:
Received 3 August 2017
Accepted 5 February 2018
Available online 8 February 2018
Submitted by J.F. Queiro

MSC:
05C50

Keywords:
Invariant factors
Elementary divisors
Smith normal form
Critical group
Jacobian group
Sandpile group
Laplacian matrix
Kneser graph

A B S T R A C T

In this paper we compute the critical group of the Kneser graph $KG(n, 2)$. This is equivalent to computing the Smith normal form of a Laplacian matrix of this graph.

☆ This work was partially supported by a grant from the Simons Foundation (#204181 to Peter Sin).
* Corresponding author.
E-mail addresses: duceyje@jmu.edu (J.E. Ducey), hillim@dukes.jmu.edu (I. Hill), sin@ufl.edu (P. Sin).

https://doi.org/10.1016/j.laa.2018.02.005
0024-3795/

1. Introduction

Given a finite graph Γ, it is of interest to compute linear algebraic invariants that arise from certain matrices attached to Γ. Two of the most common matrices are the adjacency matrix A and the Laplacian matrix L. The spectrum, for example, of the matrices does not depend on the ordering of the vertices. Another graph invariant which we will consider here is the cokernel of the matrix viewed as a map of free abelian groups. A cyclic decomposition of the cokernel is described by the Smith normal form of the matrix.

When the matrix is A, the cokernel is called the *Smith group* $S(\Gamma)$. The torsion subgroup of the cokernel of L is known as the *critical group* $K(\Gamma)$. The critical group is especially interesting; for a connected graph, the order of $K(\Gamma)$ counts the number of spanning trees of the graph. The critical group is also sometimes called the *sandpile group*, or *Jacobian group* [9].

Let $\Gamma = KG(n, 2)$, the Kneser graph with vertex set equal to the 2-element subsets of an n-element set. A pair of 2-element subsets (we will abbreviate to "2-subsets") are adjacent when they are disjoint. The complement Γ' of this graph is called the *triangular graph* $T(n)$. The Smith group $S(\Gamma')$ was computed in [3, SNF3] by using integral row and column operations on the adjacency A' to produce a diagonal form. The critical group $K(\Gamma')$ was described in [1, Cor. 9.1], where the authors study critical groups of line graphs (note that Γ' is the line graph of the complete graph on n vertices). In [10] an integral basis was found for the inclusion matrices of r-subsets vs. s-subsets that puts them in diagonal form. Observe that a pair of subsets are disjoint precisely when one is included in the complement of the other, thus the adjacency A of Γ is integrally equivalent to the 2-subsets vs. $(n-2)$-subsets inclusion matrix. In general, the Smith group of any Kneser graph can be deduced from the work [10].

In this paper we will compute the critical group $K(\Gamma)$ of the Kneser graph on 2-subsets of an n-element set. It is worth drawing attention to the quite different approaches taken in calculating the three families of groups above. In this work we will demonstrate yet another technique for calculating Smith/critical groups by applying some representation theory of the symmetric groups.

The paper is organized as follows. In Section 2 we give formal definitions and describe a few important results and notions that will be used repeatedly. In Section 3 we compute the elementary divisor decomposition of the critical group $K(\Gamma)$ by considering each prime p that divides the order of the group. In Section 4 we put this information together and obtain a concise description of the Smith normal form of L.

2. Preliminaries

2.1. Smith normal form

Let Γ be a finite graph. Under some ordering of the vertex set $V(\Gamma)$, define the adjacency matrix $A = (a_{i,j})$ by