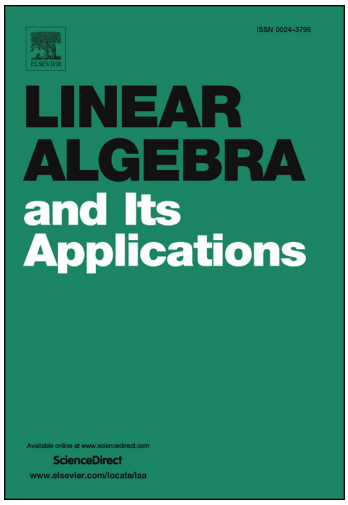

The normalized numerical range and the Davis-Wielandt shell

Brian Lins, Ilya M. Spitkovsky, Siyu Zhong

PII: S0024-3795(18)30041-7
DOI: https://doi.org/10.1016/j.laa.2018.01.027
Reference: LAA 14451

To appear in: *Linear Algebra and its Applications*

Received date: 30 November 2017
Accepted date: 21 January 2018

Please cite this article in press as: B. Lins et al., The normalized numerical range and the Davis-Wielandt shell, *Linear Algebra Appl.* (2018), https://doi.org/10.1016/j.laa.2018.01.027

The normalized numerical range and the Davis-Wielandt shell

Brian Lins[a,*], Ilya M. Spitkovsky[b], Siyu Zhong[b]

[a] *Department of Mathematics and Computer Science*
Hampden-Sydney College, Hampden Sydney VA 23943, USA
[b] *Division of Science, New York University Abu Dhabi (NYUAD)*
Saadiyat Island, P.O. Box 129188 Abu Dhabi, UAE

Abstract

For a given n-by-n matrix A, its *normalized numerical range* $F_N(A)$ is defined as the range of the function $f_{N,A}\colon x \mapsto (x^*Ax)/(\|Ax\| \cdot \|x\|)$ on the complement of $\ker A$. We provide an explicit description of this set for the case when A is normal or $n = 2$. This extension of earlier results for particular cases of 2-by-2 matrices (by Gevorgyan) and essentially Hermitian matrices of arbitrary size (by A. Stoica and one of the authors) was achieved due to the fresh point of view at $F_N(A)$ as the image of the Davis-Wielandt shell $\mathrm{DW}(A)$ under a certain non-linear mapping $h\colon \mathbb{R}^3 \mapsto \mathbb{C}$.

Keywords: Normalized numerical range, Davis-Wielandt shell, normal matrix

2010 MSC: 15A60 47A12 47B15

1. Introduction

Throughout the paper, we denote by $\mathbb{C}^n$ the standard n-dimensional inner product space over the complex field $\mathbb{C}$ and by $M_n(\mathbb{C})$ the algebra of all n-by-n matrices with entries in $\mathbb{C}$.

The classical *numerical range* $F(A)$ (a.k.a. the *field of values*, or the *Hausdorff set*) of $A \in M_n(\mathbb{C})$ is by definition the set of values of the corresponding quadratic form x^*Ax on the unit sphere $S\mathbb{C}^n := \{x \in \mathbb{C}^n : \|x\| = 1\}$ of $\mathbb{C}^n$. Equivalently,

$$F(A) = \left\{(x^*Ax)/\|x\|^2 \colon x \in \mathbb{C}^n \setminus \{0\}\right\}.$$

☆The results are partially based on the Capstone project of the third named under the supervision of the second named author. The latter was also supported in part by the Faculty Research funding from the Division of Science and Mathematics, New York University Abu Dhabi.

*Corresponding author.
Email addresses: blins@hsc.edu (Brian Lins), ims2@nyu.edu, imspitkovsky@gmail.com (Ilya M. Spitkovsky), sz1152@nyu.edu (Siyu Zhong)