Linear Algebra and its Applications 546 (2018) 210–260

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Nilpotent matrices having a given Jordan type as maximum commuting nilpotent orbit

Anthony Iarrobino [a,*], Leila Khatami [b], Bart Van Steirteghem [c], Rui Zhao [d]

[a] Department of Mathematics, Northeastern University, Boston, MA 02115, USA
[b] Department of Mathematics, Union College, Schenectady, NY 12308, USA
[c] Department of Mathematics, Medgar Evers College, City University of New York, Brooklyn, NY 11225, USA
[d] Mathematics Department, University of Missouri, Columbia, MO 65211, USA

ARTICLE INFO

Article history:
Received 29 July 2016
Accepted 3 February 2018
Available online 9 February 2018
Submitted by P. Semrl

MSC:
primary 15A27
secondary 05E40, 13E10, 15A21

Keywords:
Jordan type
Commuting nilpotent matrices
Generic commuting orbit
Nilpotent orbit
Partition

ABSTRACT

The Jordan type of a nilpotent matrix is the partition giving the sizes of its Jordan blocks. We study pairs of partitions (P, Q), where $Q = \mathfrak{Q}(P)$ is the Jordan type of a generic nilpotent matrix A commuting with a nilpotent matrix B of Jordan type P. T. Košir and P. Oblak have shown that Q has parts that differ pairwise by at least two. Such partitions, which are also known as "super distinct" or "Rogers–Ramanujan", are exactly those that are stable or "self-large" in the sense that $\mathfrak{Q}(Q) = Q$.

In 2012 P. Oblak formulated a conjecture concerning the cardinality of $\mathfrak{Q}^{-1}(Q)$ when Q has two parts, and proved some special cases. R. Zhao refined this to posit that the partitions in $\mathfrak{Q}^{-1}(Q)$ for $Q = (u, u-r)$ with $u > r > 1$ could be arranged in an $(r-1) \times (u-r)$ table $\mathcal{T}(Q)$ where the entry in the k-th row and ℓ-th column has $k + \ell$ parts. We prove this Table Theorem, and then generalize the statement to propose a Box Conjecture for the set of partitions $\mathfrak{Q}^{-1}(Q)$ for an arbitrary partition Q whose parts differ pairwise by at least two.

* Corresponding author.
E-mail addresses: a.iarrobino@neu.edu (A. Iarrobino), khatamil@union.edu (L. Khatami), bartvs@mec.cuny.edu (B. Van Steirteghem), zhaorui0408@gmail.com (R. Zhao).

https://doi.org/10.1016/j.laa.2018.02.007
0024-3795/

1. Introduction

We fix an infinite field k and denote by $\mathrm{Mat}_n(\mathsf{k})$ the ring of $n \times n$ matrices with entries in k acting on the vector space $V = \mathsf{k}^n$. Let P be a partition of n and denote by $B = J_P$ the nilpotent Jordan block matrix of partition P. Let $\mathcal{C}_B = \{A \in \mathrm{Mat}_n(\mathsf{k}) \mid AB = BA\}$ be the centralizer of B in $\mathrm{Mat}_n(\mathsf{k})$, and let $\mathcal{N}_B$ be the subvariety of nilpotent elements in $\mathcal{C}_B$.

There has been substantial work in the last ten years studying the map $\mathfrak{Q}$ that takes P to the Jordan type $\mathfrak{Q}(P)$ of a generic element of $\mathcal{N}_B$. P. Oblak conjectured a beautiful recursive description of $\mathfrak{Q}(P)$. This conjecture remains open in general (for progress on it see Section 4.1, Conjecture 4.3, Remark 4.7, and [3,6,22,25,26,34]).

An *almost rectangular* partition is one whose largest part is at most one larger than its smallest part. R. Basili introduced the invariant r_P, which is the smallest number of almost rectangular partitions whose union is P, and showed that $\mathfrak{Q}(P)$ has r_P parts (Theorem 2.4). T. Košir and P. Oblak showed that if the characteristic of k is 0 then $\mathfrak{Q}(P)$ has parts that differ pairwise by at least two (Theorem 2.6). Even in cases where the Oblak recursive conjecture had been shown some time ago, (as $r_P = 2$ [27], or $r_P = 3$ [26]) the set $\mathfrak{Q}^{-1}(Q)$ remained mysterious. In 2012 P. Oblak made a second conjecture: when $Q = (u, u - r)$ with $u > r \geq 2$, then the cardinality $|\mathfrak{Q}^{-1}(Q)| = (r - 1)(u - r)$ [35, Remark 2]. In 2013, R. Zhao noticed an even stronger pattern in $\mathfrak{Q}^{-1}(Q)$ for such Q. She conjectured that there is a table $\mathcal{T}(Q)$ of partitions $P_{k,\ell}$ where the number of parts in $P_{k,\ell}$ is $k + \ell$: see Theorem 1.1 immediately below. We here prove a precise version, the Table Theorem (Theorems 3.12 and 3.19). We then propose a Box Conjecture 4.11 describing $\mathfrak{Q}^{-1}(Q)$ for arbitrary partitions Q whose parts differ pairwise by at least two (Section 4.2), and we study some special cases where Q has three parts (Section 4.3).

The question, which pairs of conjugacy classes can occur for pairs of commuting matrices reduces to the case where both matrices are nilpotent. There is an extensive literature on commuting pairs of nilpotent matrices, including [3,19,22,25–27,34–36,39] and others, some of whose results we specifically cite. Connections to the Hilbert scheme are made in [1,2,4,10,20,31,39], and commuting nilpotent orbits occur in the study of Artinian algebras [4,21]. However, the study of the map $P \to \mathfrak{Q}(P)$ seems to be, surprisingly, very recent, beginning with [1,2,4,25,27,34,36,39]: apparently, early workers in the area were more drawn to determining vector spaces of commuting matrices of maximum dimension (see [24,29,42] and references in the latter). There is further recent work on commuting r-tuples of nilpotent matrices, as [19,33,41], and these also appear to be connected to the study of group schemes [14,32,43,44]. There is much study of nilpotent orbits for Lie algebras, as in [9,11,16,37]; for generalizations of problems considered here to other Lie algebras than sl_n, see [36].

Our main result is

Theorem 1.1. *Let $Q = (u, u - r)$ where $u > r \geq 2$.*

i. *The cardinality $|\mathfrak{Q}^{-1}(Q)| = (r - 1)(u - r)$.*