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On the non-existence of antipodal cages of even girth

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ABSTRACT

The Moore bound $M(k,g)$ is a lower bound on the order of k-regular graphs of girth g (denoted (k,g)-graphs). The excess e of a (k,g)-graph of order n is the difference $e = n - M(k,g)$. A (k,g)-cage is a (k,g)-graph with the fewest possible number of vertices. A graph of diameter d is said to be antipodal if, for any vertices u, v, w such that $d(u,v) = d$ and $d(u,w) = d$, it follows that $d(v,w) = d$ or $v = w$. Biggs and Ito proved that any (k,g)-cage of even girth $g = 2d \geq 6$ and excess $e \leq k-2$ is a bipartite graph of diameter $d+1$. In this paper we treat (k,g)-cages of even girth and excess $e \leq k-2$. Based on spectral analysis we give a relation between the eigenvalues of the adjacency matrix A and the distance matrix A_{d+1} of such cages. Applying matrix theory, we prove the non-existence of antipodal (k,g)-cages of excess e, for $k \geq e+2 \geq 4$ and $g = 2d \geq 14$.

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1. Introduction

A (k, g)-*graph* is a k-regular graph having girth g. A (k, g)-*cage* is a (k, g)-graph of a smallest order. The *Cage Problem* or *Degree/Girth Problem* calls for finding cages; Tutte was the first to study (k, g)-cages [17]. A (k, g)-graph exists for any pair (k, g), where $k \geq 2$ and $g \geq 3$, see [8] and [15]. It is well known that (k, g)-graphs have at least $M(k, g)$ vertices, where

$$M(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2}, & g \text{ odd}, \\ 2\left(1 + (k-1) + \cdots + (k-1)^{(g-2)/2}\right), & g \text{ even}. \end{cases} \quad (1)$$

If G is a (k, g)-graph of order n, then we define the *excess* e of G to be $n - M(k, g)$. The graphs whose orders are equal to $M(k, g)$ (graphs of excess 0) are called *Moore graphs*. Their classification has been completed except for the case $k = 57$ and $g = 5$. The Moore graphs exist if $k = 2$ and $g \geq 3$, $g = 3$ and $k \geq 2$, $g = 4$ and $k \geq 2$, $g = 5$ and $k = 2, 3, 7$, or $g = 6, 8, 12$ and a generalized n-gon of order $k - 1$ exists, see [1], [7] and [9].

The following three results concern the graphs of even girth.

Theorem 1.1 *(Biggs and Ito [4]). Let G be a (k, g)-cage of girth $g = 2d \geq 6$ and excess e. If $e \leq k - 2$, then e is even and G is bipartite of diameter $d + 1$.*

It is known that these graphs are partially distance-regular. To learn more about almost-distance-regular graphs, see [5]. For the next theorem, let $D(k, 2)$ denote the incidence graph of a symmetric $(v, k, 2)$-design.

Theorem 1.2 *(Biggs and Ito [4]). Let G be a (k, g)-cage of girth $g = 2d \geq 6$ and excess 2. Then $g = 6$, G is a double-cover of $D(k, 2)$, and $k \not\equiv 5, 7 \pmod 8$.*

The following result is based on a divisibility criterion obtained through counting g-cycles in a (k, g)-cage of excess 4.

Theorem 1.3 *(Jajcayová, Filipovski and Jajcay [12]). Let $k \geq 6$ and $g = 2d > 6$. No (k, g)-graphs of excess 4 exist for parameters k, g satisfying at least one of the following conditions:*

1) *$g = 2p$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod p$;*
2) *$g = 4 \cdot 3^s$ such that $s \geq 4$, and k is divisible by 9 but not by 3^{s-1};*
3) *$g = 2p^2$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2 \pmod p$ and k even;*
4) *$g = 4p$, with $p \geq 5$ a prime number, and $k \not\equiv 0, 1, 2, 3, p - 2 \pmod p$;*
5) *$g \equiv 0 \pmod{16}$, and $k \equiv 3 \pmod g$.*

Let $k \geq 4$, $g = 2d \geq 6$ and let G be a (k, g)-cage of excess $e \leq k - 2$ and order n. Due to Theorem 1.1, we conclude that G is a bipartite graph of diameter $d + 1$. Let A be its