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On line graphs with maximum energy

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For an undirected simple graph G, the line graph $\mathcal{L}(G)$ is the graph whose vertex set is in one-to-one correspondence with the edge set of G where two vertices are adjacent if their corresponding edges in G have a common vertex. The energy $\mathcal{E}(G)$ is the sum of the absolute values of the eigenvalues of G. The vertex connectivity $\kappa(G)$ is referred as the minimum number of vertices whose deletion disconnects G. The positive inertia $\nu^+(G)$ corresponds to the number of positive eigenvalues of G. Finally, the matching number $\beta(G)$ is the maximum number of independent edges of G. In this paper, we derive a sharp upper bound for the energy of the line graph of a graph G on n vertices having a vertex connectivity less than or equal to k. In addition, we obtain upper bounds on $\mathcal{E}(G)$ in terms of the edge connectivity, the inertia and the matching number of G. Moreover, a new family of hyperenergetic graphs is obtained.

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1. Introduction

Let G be an undirected simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. A graph G is bipartite if there exists a partitioning of $V(G)$ into disjoint, nonempty sets V_1 and V_2 such that the end vertices of each edge in G are in distinct sets V_1, V_2. In this case V_1, V_2 are referred as a bipartition of G. A graph G is a complete bipartite graph if G is bipartite with bipartition V_1 and V_2 where each vertex in V_1 is connected to all the vertices in V_2. If G is a complete bipartite graph and $|V_1| = p$ and $|V_2| = q$ the graph G is written $K_{p,q}$. The Laplacian matrix of G is the $n \times n$ matrix $L(G) = D(G) - A(G)$ where $A(G)$ and $D(G)$ are the adjacency matrix and the diagonal matrix of vertex degrees of G [17,18,32], respectively. It is well known that $L(G)$ is a positive semi-definite matrix and that $(0, e)$ is an eigenpair of $L(G)$ where e is the all ones vector. The matrix $Q(G) = A(G) + D(G)$ is called the signless Laplacian matrix of G [8–10]. The eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ are called the eigenvalues, Laplacian eigenvalues and signless Laplacian eigenvalues of G, respectively. The matrices $Q(G)$ and $L(G)$ are positive semidefinite, see [38]. The spectra of $L(G)$ and $Q(G)$ coincide if and only if G is a bipartite graph, see [5,8,17,18]. Let

$$\begin{aligned} &\lambda_n(G) \le \lambda_{n-1}(G) \le \cdots \le \lambda_1(G), \\ &0 \le q_n(G) \le q_{n-1}(G) \le \cdots \le q_1(G), \quad \text{and} \\ &0 = \mu_n(G) \le \mu_{n-1}(G) \le \cdots \le \mu_1(G) \end{aligned}$$

be the eigenvalues of $A(G)$, $Q(G)$ and $L(G)$, respectively. The multiplicity of 0 as a signless Laplacian eigenvalue of a graph G without isolated vertices corresponds to the number of bipartite components of G, see [3]. The line graph $\mathcal{L}(G)$ is the graph whose vertex set is in one-to-one correspondence with the edge set of G where two vertices are adjacent if and only if the corresponding edges in G have a common vertex [24]. The energy of the line graph of a graph G and its relations with the other graph energies were earlier studied in [12,21]. The spectral radius of $Q(G)$ is called the signless Laplacian index of G and it is usually denoted by $q_1(G)$. From Perron–Frobenius Theory for nonnegative matrices, it follows that if G is a connected graph then $q_1(G)$ is a simple eigenvalue of $Q(G)$.

We recall the notion of the join operation of graphs. Given two vertex disjoint graphs G_1 and G_2, the join of G_1 and G_2 is the graph $G = G_1 \vee G_2$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$.

The join operation of two vertex disjoint graphs can be generalized as follows [6,7]. Let H be a graph of order k. Let $V(H) = \{1, 2, ..., k\}$. Let $\mathcal{F} = \{G_1, G_2, ..., G_k\}$ be a set of pairwise vertex disjoint graphs. Each vertex $j \in V(H)$ is assigned to the graph $G_j \in \mathcal{F}$. Let G be the graph obtained from the graphs $G_1, G_2, ..., G_k$ and the edges connecting each vertex of G_i with all the vertices of G_j for all edge $ij \in E(H)$. That is, G is the graph with vertex set