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Bounds related to Coxeter spectral measures of graphs

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ABSTRACT

Let M be a real square matrix. We give upper bounds for the sum of the absolute values of the (real part of the) eigenvalues of M, quantity in some particular cases known as (real) energy of M. From these results we obtain a combinatorial bound for the real energy of the Coxeter matrix Φ_Q of a tree digraph Q with n vertices,

$$e_{re}(\Phi_Q) \leq \mathbf{min}\sqrt{\frac{n(2a+b+c+d)}{2}},$$

where a is the number of edges, b and c are respectively the number of bifurcation and congregation paths of Q (as defined below), $d = \sum_{i=1}^{n}[\delta(i)-1][\delta(i)-2]$ with $\delta(i)$ the degree of a vertex i, and where the minimum is taken over all possible orientations of edges in Q. As particular case we consider Dynkin, Euclidean and star graphs, obtaining practical bounds for the real Coxeter energy of these classes of digraphs.

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Mahler measure
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1. Introduction

Spectral properties of Coxeter transformations have attracted attention in fields of mathematics such as graph theory, Lie algebras and representation theory of associative algebras for a long time. Classical results and methods in linear algebra and matrix theory have provided deeper insight into these fields, see for instance [4], [16], [17], [25] and [20] to name a few references on representation theory. Some of these techniques have applications in a variety of disciplines such as chemistry, data analysis and probability theory, where plenty of research has been conducted (cf. [1], [5], [22], [13] and references therein).

The purpose of this paper is to incorporate and extend some old and new results in matrix theory to the study of Coxeter invariants related to tree graphs (and their associated hereditary algebras). To be precise, we investigate bounds for Coxeter measures as energy (classical, real and Nikiforov energies), Mahler measure and matrix norms. These notions have diverse origins, from the approximations of energies of molecular structures in chemistry to graph and number theory. A special feature of our work is the comparison of eigenvalues and singular values of Coxeter (and related) matrices, with help of the *majorization relation* for real vectors (see 2.3 below).

One of our interests is the so-called *(real) Coxeter energy* of a (di)graph Q: the sum of absolute values of the (real part of) eigenvalues of Φ_Q, the *Coxeter matrix* of Q. In the Main Results we give general upper bounds for these measures (Theorems 2.3 and 3.2), and use them to derive a combinatorial bound for the real Coxeter energy of a tree digraph (Corollary 4.3). As brief application we give practical bounds for *star digraphs* (Proposition 5.3). Closely related topics in a wider context have recently appeared in published papers as [11] and [14], to mention a couple of references, together with a new computational approach to Coxeter spectral analysis and applications to the study of finite dimensional algebras. This is a sample of the lively area of spectral analysis in the Coxeter formalism.

The Coxeter energy was introduced in [18] by the first named author for finite dimensional algebras, and it was later studied for graphs and digraphs in [13]; the real Coxeter energy was defined in [13] where the author describes an integral formula of Coulson type for computing its value. In Section 2 we establish some general bounds for Coxeter energies, and comment on their relation to other classical concepts. Sections 3 and 4 are devoted to the study of the Coxeter real energy, where we find bounds in terms of combinatorial descriptions of quadratic traces of the Coxeter matrix. In Section 5 we apply these results to give bounds for star digraphs, among other related results.