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Up and down Grover walks on simplicial complexes

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ABSTRACT

Notions of up and down Grover walks on simplicial complexes are proposed and their properties are investigated. These are abstract Szegedy walks, which is a special kind of unitary operators on a Hilbert space. The operators introduced in the present paper are usual Grover walks on graphs defined by using combinatorial structures of simplicial complexes. But the shift operators are modified so that it can contain information of orientations of each simplex in the simplicial complex. It is well-known that the spectral structures of this kind of unitary operators are almost determined by its discriminant operators. It has strong relationship with combinatorial Laplacian on simplicial complexes and geometry, even topology, of simplicial complexes. In particular, theorems on a relation between spectrum of down discriminants and orientability, on a relation between symmetry of spectrum of discriminants and combinatorial structure of simplicial complex are given. Some examples, both of finite and infinite simplicial complexes, are also given. Finally, some aspects of finding probability and stationary measures are discussed.

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1. Introduction

Grover walks, originally introduced in [1] and named after a famous work of Grover [2] on a quantum search algorithm, is one of unitary time evolution operators, often called discrete-time quantum walks, defined over graphs. These are introduced in computer sciences and developed in areas of mathematics, such as probability theory, spectral theory and geometric analysis. It was Szegedy [3] who had realized that their spectral structure of *Szegedy walks*, which generalizes Grover walks, are almost determined by a self-adjoint operator, called *discriminant* operator. Szegedy's idea also works well for infinite graphs as is developed in [4,5]. More concretely, an abstract Szegedy walk, is a unitary operator of the form

$$U = SC, \tag{1}$$

where S and C are unitary operators on a separable Hilbert space satisfying $S^2 = C^2 = I$.

In [6], certain class of unitary transitions on simplicial complexes are introduced. Suppose that $\mathcal{K} = (V, \mathcal{S})$ is a simplicial complex with certain conditions where V is a set of vertices and $\mathcal{S}$ is a set of simplices. Let $\widehat{K}_q$ be the set of sequences of vertices of length $q+1$ which form simplices in $\mathcal{S}$. The symmetric group $\mathfrak{S}_{q+1}$ of order $(q+1)!$ acts on $\widehat{K}_q$ naturally. The operators introduced in [6] act on the Hilbert space $\ell^2(\widehat{K}_q)$. They have the form (1), but the operator S, which is often called a 'shift operator' of an abstract Szegedy walk, is given by the action of certain permutation $\pi \in \mathfrak{S}_{q+1}$, and hence in general it does not satisfy $S^2 = I$. It seems that the operators introduced in [6] would have rather advantage, because one can choose permutations π for various purposes. However, to find their geometric aspects, it does not seem so transparent, because it is not quite clear which permutation should be chosen to relate the operators with geometry. Simplicial complex is a geometric, topological and combinatorial object. Hence it would be rather natural to expect that operators so-defined have geometric information. Like Laplacians acting on differential forms, there is a notion of *combinatorial Laplacians* which is defined by replacing the exterior differentials in the definition of Laplacians acting on differential forms by the coboundary operator in simplicial cohomology theory. A general framework for this combinatorial Laplacian was investigated in an interesting article [7]. They have a rich geometric aspects, such as Hodge decomposition.

The purpose in the present paper is to introduce and investigate other Grover walks on simplicial complexes. The definition is rather simple. The operators we mainly consider are Grover walks on graphs, which we call *up and down graphs* (and it is essentially the same as the dual graph used in [7]), defined by using combinatorial structures of simplicial complexes. They are basically Grover walks on graphs, but the shift operator is a bit different. Namely it is modified from the usual shift operators on graphs, which will be necessary to take the orientation of simplices into account. Indeed this modification makes Grover walks on up and down graphs, which we call *up and down Grover walks*, certainly have geometric aspects. We also consider the alternating sum of the operators