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Identities with involution for 2×2 upper triangular matrices algebra over a finite field

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A B S T R A C T

Let $UT_2(F)$ be the 2×2 upper triangular matrices algebra over a finite field F of characteristic different from 2. For every involution of the first kind of $UT_2(F)$ we describe the set of all $*$-polynomial identities for this algebra.

1. Introduction

Let F be a field and let $UT_n(F)$ be the algebra of $n \times n$ upper triangular matrices over F. This algebra plays an important role in the theory of PI-algebras. Maltsev was

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the first to describe the set of polynomial identities of $UT_n(F)$, denoted by $Id(UT_n(F))$. Before to state his result, let $F\langle X\rangle$ be the free unitary associative algebra freely generated by the infinite set $X = \{x_1, x_2, \ldots\}$. Maltsev proved in [3] that if the characteristic of F is $\mathrm{char}(F) = 0$ then $Id(UT_n(F))$ is generated, as a T-ideal of $F\langle X\rangle$, by the polynomial

$$[x_1, x_2][x_3, x_4]\cdots[x_{2n-1}, x_{2n}],$$

where $[x, y] = xy - yx$. The same is hold when F is infinite. If F is an arbitrary field (finite or infinite) then

$$Id(UT_n(F)) = [Id(UT_1(F))]^n.$$

This result was proved by Siderov in [4]. Note that $UT_1(F) = F$. It's known that $Id(F)$ is generated as a T-ideal by

a) $[x_1, x_2]$ if F is infinite,

b) $[x_1, x_2]$ and $x_1^q - x_1$ if F is finite with q elements.

Using Jacobson's theorem we can omit $[x_1, x_2]$ in item b). See [2, Exercise 2.3.6] and [1, Theorem 5.78] for details.

Let G be a group. Valenti and Zaicev described the G-gradings on $UT_n(F)$. They proved in [5] that every G-grading on $UT_n(F)$ is elementary. In [6] Vincenzo, Koshlukov and Valenti described the G-graded polynomial identities of $UT_n(F)$ if F is infinite.

Let F be a field of $\mathrm{char}(F) \neq 2$. If $UT_n(F)$ has an involution, denote by $Id(UT_n(F), *)$ its $*$-polynomial identities. Di Vincenzo, Koshlukov and La Scala described the involutions of first kind of $UT_n(F)$. They proved in [7] that there exist two classes of inequivalent involutions when n is even and a single class otherwise. They also described:

a) $Id(UT_2(F), *)$ when F is infinite,

b) $Id(UT_3(F), *)$ when $\mathrm{char}(F) = 0$,

for all involutions of first kind on $UT_2(F)$ and $UT_3(F)$ respectively. It's an open problem describe $Id(UT_n(F), *)$ in other cases.

In this paper we describe $Id(UT_2(F), *)$ for all involutions of first kind of $UT_2(F)$ when F is finite.

2. Some properties of finite fields

We will use several times the next two lemmas:

Lemma 2.1. *[2, Proposition 4.2.3] Let F be a finite field, $|F| = q$. Then the monomials*

$$x_1^{d_1}\cdots x_n^{d_n},\ 0 \le d_1, \ldots, d_n \le q - 1,$$

are linearly independent modulo $Id(F)$.