Linear Algebra and its Applications 544 (2018) 273–285

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Principal eigenvectors and spectral radii of uniform hypergraphs

Haifeng Li [a], Jiang Zhou [b], Changjiang Bu [a,b,*]

[a] *College of Automation, Harbin Engineering University, Harbin 150001, PR China*
[b] *College of Science, Harbin Engineering University, Harbin 150001, PR China*

A R T I C L E   I N F O

Article history:
Received 1 August 2016
Accepted 11 January 2018
Submitted by R. Brualdi

MSC:
05C50
05C65
15A69
15A18

Keywords:
Hypergraph
Spectral radius
Principal eigenvector

A B S T R A C T

In this paper, some inequalities among the principal eigenvector, spectral radius and vertex degrees of a connected uniform hypergraph are established. Necessary and sufficient conditions of equalities holding are presented, which are related to the regularity of a hypergraph. Furthermore, we present some bounds on the spectral radius for a connected irregular uniform hypergraph in terms of some parameters, such as principal ratio, maximum degree, diameter, and the number of vertices and edges.

1. Introduction

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$. An order m dimension n tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a multidimensional array with n^m entries, where $i_j \in [n]$, $j \in [m]$. When $m = 2$, $\mathcal{A}$ is an $n \times n$ matrix. Let $\mathbb{C}^{[m,n]}$ be the set of order m dimension n tensors

* Corresponding author.
E-mail address: buchangjiang@hrbeu.edu.cn (C. Bu).

https://doi.org/10.1016/j.laa.2018.01.017

over the complex field $\mathbb{C}$, and $\mathbb{C}^n$ be the set of n-vectors over the complex field $\mathbb{C}$. For $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$, if all the entries $a_{i_1 i_2 \cdots i_m} \geq 0$, then $\mathcal{A}$ is called nonnegative.

In 2005, Qi [22] and Lim [14] defined the eigenvalues of tensors, respectively. For $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ and $x = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$, $\mathcal{A}x^{m-1}$ is an n-vector whose the i-th component is

$$\left(\mathcal{A}x^{m-1}\right)_i = \sum_{i_2,\ldots,i_m=1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.$$

If there exists a number $\lambda \in \mathbb{C}$ and a nonzero vector $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$, x is called an eigenvector of $\mathcal{A}$ corresponding to λ, where $x^{[m-1]} = \left(x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}\right)^{\mathrm{T}}$. The spectral radius $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$, where $\sigma(\mathcal{A})$ is the set of all eigenvalues of $\mathcal{A}$.

A hypergraph G is a pair $(V(G), E(G))$, where $E(G) \subseteq P(V(G))$ and $P(V(G))$ stands for the power set of $V(G)$. The elements of $V(G)$ are called the vertices and the elements of $E(G)$ are called the edges (see [1]). If each edge of G contains exactly k distinct vertices, then G is called k-uniform. When $k = 2$, G is a graph. For all $i \in V(G)$, $E_i(G)$ denotes the set of edges containing i, and $d_i = |E_i(G)|$ denotes the degree of i, $\Delta = \max_i\{d_i\}$ and $\delta = \min_i\{d_i\}$. If $\Delta = \delta$, then G is called regular. The adjacency tensor [6] of a k-uniform hypergraph G, denoted by $\mathcal{A}_G$, is an order k dimension $|V(G)|$ nonnegative tensor with entries

$$a_{i_1 i_2 \cdots i_k} = \begin{cases} \dfrac{1}{(k-1)!}, & \textit{if } \{i_1, i_2, \ldots, i_k\} \in E(G), \\ 0, & \textit{otherwise}. \end{cases}$$

Eigenvalues of $\mathcal{A}_G$ are called eigenvalues of G, the spectral radius of $\mathcal{A}_G$ is called the spectral radius of G, denoted by $\rho(G)$.

For k-uniform hypergraph G with n vertices, it is connected if and only if $\mathcal{A}_G$ is nonnegative weakly irreducible [8,21,27]. By the Perron–Frobenius theorem [27], $\rho(G)$ is an eigenvalue of G and there exists a unique positive eigenvector $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ corresponding to $\rho(G)$ with $\sum_{i=1}^{n} x_i^k = 1$, which is called the principal eigenvector of G. The maximum and minimum entries of x are denoted by $x_{\max}$ and $x_{\min}$, respectively. The parameter $\gamma = \frac{x_{\max}}{x_{\min}}$ is called the principal ratio of G (see [17]). For $e = \{i_1, i_2, \ldots, i_k\} \in E(G)$, let $x^e = \prod_{j=1}^{k} x_{i_j}$ and $x^{e \backslash \{i_1\}} = \prod_{j=2}^{k} x_{i_j}$.

The principal eigenvector of a connected graph is one of centrality metrics [11]. Let G be a connected irregular graph. Investigations have been conducted to the relationship between $x_{\max}$ and the structure of a connected graph [20], bounds on γ, $x_{\max}$, $x_{\min}$