Linear Algebra and its Applications 544 (2018) 299–305

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

On the rank of a Latin tensor

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ARTICLE INFO

Article history:
Received 23 September 2016
Accepted 13 January 2018
Submitted by L.-H. Lim

MSC:
15A03
15A69

Keywords:
Tensor rank
Latin square

ABSTRACT

An $n \times n$ matrix A is called permutative if the rows of A are distinct permutations of a family of n distinct elements. For all $n \geqslant 3$, we show that the minimal rank of a non-negative permutative matrix equals 3. The minimal rank of a generic permutative $n \times n$ matrix equals the smallest integer r such that $r! \geqslant n$. Our results answer the questions asked recently by Hu, Johnson, Davis, Zhang, and we show how to generalize them to tensors.

An $m \times n$ matrix A is called permutative if the rows of A are distinct permutations of a family of n distinct elements. In this paper, we study the minimal ranks of permutative matrices with elements taken in different sets. The permutative matrices are a further generalization of the classes of circulants and Latin squares, and the minimum rank problems for these classes have been studied previously in different contexts, see the papers [6,11] and references therein.

In particular, we are interested in real permutative matrices and generic permutative matrices. Here, the word 'generic' means that a matrix consists of variables $x_1, \ldots, x_n$, and the rank is taken with respect to the field $\mathbb{R}(x_1, \ldots, x_n)$. Alternatively, the generic

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https://doi.org/10.1016/j.laa.2018.01.018

rank of such a matrix is the rank of almost all matrices obtained from it by replacing the variables with real numbers. One of our results is the following.

Theorem 1. *For $n \geqslant 2$, the minimal rank of a generic permutative $n \times n$ matrix equals the smallest integer r satisfying $r! \geqslant n$.*

In order to formulate one more result, we need to recall some basic definitions of multilinear algebra. The concept of a *Latin tensor* has been studied under the names *Latin box* and *Latin hypercube*, see [1,7], but we prefer to call it a tensor because we are interested in tensor ranks. A *Latin tensor* is a d-dimensional $n \times \ldots \times n$ array L filled with distinct elements $x_1, \ldots, x_n$ such that, for all fixed $m_1, \ldots, m_{k-1}, m_{k+1}, \ldots, m_n$, we have

$$\{L_{m_1,\ldots,m_{k-1},\widetilde{m_k},m_{k+1},\ldots,m_n} \mid \widetilde{m_k} = 1, \ldots, n\} = \{x_1, \ldots, x_n\}. \tag{1}$$

In what follows, the size of a d-dimensional $n \times \ldots \times n$ tensor is denoted by $n^{\times d}$ to keep the notation concise. We recall that an $n^{\times d}$ tensor T is called a *rank-one* tensor if there are numbers (a_{ij}) with $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$ such that

$$T_{m_1,\ldots,m_d} = a_{1m_1} \cdot \ldots \cdot a_{dm_d},$$

for all indexing tuples $(m_1, \ldots, m_d)$. The *rank* of a tensor T is the smallest k such that T can be written as a sum of k rank-one tensors. The coordinates of all tensors in our paper belong to $\mathbb{C}$, and we compute their ranks with respect to the field of complex numbers.

If $d = 2$, then tensors become matrices, and the rank of a tensor corresponds to the usual matrix rank. More than that, the rank of a matrix does not depend on the field over which it is computed, so we identify the complex and real ranks in the case of real matrices. (We note in passing that there exist higher order tensors whose real and complex ranks are different, see the paper [2] for details.) When $d = 2$, we also see that the Latin tensors are exactly the *Latin squares*, which means that they are permutative matrices whose transposes are also permutative. Another result that we obtain in our paper is as follows.

Theorem 2. *For all d, n, there is a Latin $n^{\times d}$ tensor with non-negative real entries that has rank at most 3.*

In the special case $d = 2$, this theorem leads to the solution of the problem posed by Hu, Johnson, Davis, Zhang. Namely, we show that every $n \times n$ permutative non-negative matrix has rank at least 3 whenever $n \geqslant 3$, which allows us to state the following result.

Theorem 3. *The minimal rank of a non-negative permutative $n \times n$ matrix equals*