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# Signless Laplacian energy of a graph and energy of a line graph ☆

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ABSTRACT

For a simple graph $G$ of order $n$, size $m$ and with signless Laplacian eigenvalues $q_1, q_2, \ldots, q_n$, the signless Laplacian energy $QE(G)$ is defined as $QE(G) = \sum_{i=1}^{n} |q_i - \overline{d}|$, where $\overline{d} = \frac{2m}{n}$ is the average vertex degree of $G$. We obtain the lower bounds for $QE(G)$, in terms of first Zagreb index $M_1(G)$, maximum degree $d_1$, second maximum degree $d_2$, minimum degree $d_n$ and second minimum degree $d_{n-1}$. As a consequence of these bounds, we obtain several bounds for the energy $E(\mathscr{L}(G))$ of the line graph $\mathscr{L}(G)$ of graph $G$ in terms of various graph parameters like $M_1(G)$, $\omega$ (the clique number), $n$, $m$, etc., which improve some recently known bounds.



## 1. Introduction

Let $G(V, E)$ be a simple graph with $n$ vertices, $m$ edges with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. In case the graph $G$ is to be

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mentioned, the number of vertices and edges are respectively denoted by $n(G)$ and $m(G)$. The adjacency matrix $A = (a_{ij})$ of $G$ is a $(0,1)$-square matrix of order $n$ whose $(i,j)$-entry is equal to 1, if $v_i$ is adjacent to $v_j$ and equal to 0, otherwise. The spectrum of the adjacency matrix is called the adjacency spectrum of $G$. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the adjacency eigenvalues of $G$, the energy [13] of $G$ is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. This quantity introduced by I. Gutman has noteworthy chemical applications and its mathematical aspect is well developed (see [17]).

Let $D(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to $G$, where $d_i = d(v_i) = \deg(v_i)$, for all $i = 1, 2, \ldots, n$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively, called the Laplacian and the signless Laplacian matrices and their spectrum are respectively, called the Laplacian spectrum and the signless Laplacian spectrum of the graph $G$. Both the matrices $L(G)$ and $Q(G)$ are real symmetric, positive semi-definite matrices, therefore their eigenvalues are non-negative real numbers. Let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ be the Laplacian spectrum and signless Laplacian spectrum of $G$, respectively. It is well known that $\mu_n = 0$ with multiplicity equal to the number of connected components of $G$, and $\mu_{n-1} > 0$ if and only if $G$ is connected. The positive inertia of a graph $G$ is the number of positive adjacency eigenvalues of $G$ and its negative inertia is the number of negative adjacency eigenvalues. The sum of positive and negative inertia of $G$ is always equal to the rank of $G$.

The motivation for Laplacian energy comes from graph energy [13,17]. The Laplacian energy of a graph $G$ as put forward by Gutman and Zhou (see [15]) is defined as $LE(G) = \sum_{i=1}^{n} \left|\mu_i - \frac{2m}{n}\right|$. This quantity, which is an extension of graph-energy concept, has found remarkable chemical applications beyond the molecular orbital theory of conjugated molecules (see [24]). For recent development on $LE(G)$ see [6,10,11,21,22] and the references therein. It is easy to see that

$$tr(L(G)) = \sum_{i=1}^{n} \mu_i \quad = 2m = \quad \sum_{i=1}^{n} q_i = tr(Q(G)),$$

where $tr$ is the trace.

In analogy to Laplacian energy, the signless Laplacian energy $QE(G)$ of $G$ is defined as $QE(G) = \sum_{i=1}^{n} \left|q_i - \frac{2m}{n}\right|$. If $\sigma$, $1 \leq \sigma \leq n-1$, are the number of signless Laplacian eigenvalues greater than or equal to average degree $\overline{d} = \frac{2m}{n}$, then by a simple observation, it follows that

$$QE(G) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}| = 2S_\sigma^+(G) - \frac{4m\sigma}{n} = \max_{1 \leq i \leq n-1} \left\{ 2S_i^+(G) - \frac{4mi}{n} \right\}, \tag{1}$$

where $S_\sigma^+(G) = \sum_{i=1}^{\sigma} q_i$ is the sum of $\sigma$ largest signless Laplacian eigenvalues of $G$.