Linear Algebra and its Applications 544 (2018) 350–369

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Gram determinants of real binary tensors

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ARTICLE INFO

Article history:
Received 15 December 2016
Accepted 13 January 2018
Submitted by L.-H. Lim

MSC:
15A69
12D15
14P10

Keywords:
Tensors
Sum-of-squares
Semi-algebraic geometry
Singular value decomposition

ABSTRACT

A binary tensor consists of 2^n entries arranged into hypercube format $2 \times 2 \times \cdots \times 2$. There are n ways to flatten such a tensor into a matrix of size $2 \times 2^{n-1}$. For each flattening, M, we take the determinant of its Gram matrix, $\det(MM^T)$. We consider the map that sends a tensor to its n-tuple of Gram determinants. We propose a semi-algebraic characterization of the image of this map. This offers an answer to a question raised by Hackbusch and Uschmajew concerning the higher-order singular values of tensors.

1. Introduction

The Gram determinants of a real binary tensor of format $2 \times 2 \times \cdots \times 2$ (n times) are an n-tuple of quadratic invariants of the tensor. We introduce the *Gram locus*, the locus of tuples that arise as the Gram determinants of a real binary tensor. Here, the Gram locus is equal to the "set of feasible higher-order singular values", from [8], under change of coordinates. The Gram determinants offer a convenient set of coordinates for studying the higher order singular values of a tensor.

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https://doi.org/10.1016/j.laa.2018.01.019
0024-3795/

In Theorem 1.3 we find the convex hull of the Gram locus for real binary tensors. It is a convex polytope that we describe explicitly. Its facet defining inequalities are that each Gram determinant is bounded by the sum of the others. We give a sum-of-squares proof. In Theorem 1.4, we express the Gram locus as a semi-algebraic set for the case of $2 \times 2 \times 2$ tensors. The semi-algebraic description determines whether a tuple lies in the Gram locus or its complement, and characterizes tuples on the boundary. The non-linear part of the boundary of the Gram locus is Fig. 4, and it is depicted in the highest higher order singular value coordinates in Fig. 6.

In Section 1 of [8] it is conjectured that tensors with strictly decreasing and positive singular values in each flattening lie in the interior of the feasible set. Example 3.1 is a counter-example to this conjecture. It is a tensor on the boundary of the feasible set whose higher order singular values in each flattening are distinct and positive. Its singular values are located at the black dot in Fig. 6.

Conjecture 1.5 proposes the general form for the Gram locus. It has a concise expression as the non-negativity of a single polynomial in the Gram determinants.

Finally, Section 4 gives a partial answer to [8, Problem 1.6], characterizing the tensors whose higher order singular values coincide. In the case of matrices agreement of singular values implies orthogonal equivalence, but this is not true for tensors. Theorem 4.1 shows that the hyperdeterminant bridges the gap between orthogonal equivalence of tensors and agreement of the higher order singular value decomposition in the $2 \times 2 \times 2$ case. Our set-up for the $2 \times 2 \times 2$ tensor format is described in the following example.

Example 1.1. The $2 \times 2 \times 2$ tensor (a_{ijk}), $0 \leq i, j, k \leq 1$, has eight entries which populate the vertices of the three-cube. It has three flattenings, each of size 2×4:

$$\begin{bmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \end{bmatrix} \quad \begin{bmatrix} a_{000} & a_{001} & a_{100} & a_{101} \\ a_{010} & a_{011} & a_{110} & a_{111} \end{bmatrix} \quad \begin{bmatrix} a_{000} & a_{010} & a_{100} & a_{110} \\ a_{001} & a_{011} & a_{101} & a_{111} \end{bmatrix}.$$

For the ith flattening M we find $d_i := \det(MM^T)$. For instance, the first Gram determinant is

$$\begin{aligned} d_1 = {} & (a_{000}a_{101} - a_{001}a_{100})^2 + (a_{000}a_{110} - a_{010}a_{100})^2 + (a_{000}a_{111} - a_{011}a_{100})^2 + \\ & (a_{001}a_{110} - a_{010}a_{101})^2 + (a_{001}a_{111} - a_{011}a_{101})^2 + (a_{010}a_{111} - a_{011}a_{110})^2. \end{aligned}$$

A computation reveals that the linear combination $d_2 + d_3 - d_1$ can be written as a sum of three squared terms:

$$\begin{aligned} & 2(a_{000}a_{011} - a_{010}a_{001})^2 + 2(a_{100}a_{111} - a_{110}a_{101})^2 + \\ & (a_{010}a_{101} + a_{001}a_{110} - a_{011}a_{100} - a_{000}a_{111})^2. \end{aligned}$$

The sum-of-squares certificate certifies that the expression is non-negative for all real values of the variables.