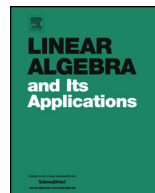




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Gram determinants of real binary tensors



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ABSTRACT

A binary tensor consists of 2^n entries arranged into hypercube format $2 \times 2 \times \cdots \times 2$. There are n ways to flatten such a tensor into a matrix of size $2 \times 2^{n-1}$. For each flattening, M , we take the determinant of its Gram matrix, $\det(MM^T)$. We consider the map that sends a tensor to its n -tuple of Gram determinants. We propose a semi-algebraic characterization of the image of this map. This offers an answer to a question raised by Hackbusch and Uschmajew concerning the higher-order singular values of tensors.

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1. Introduction

The Gram determinants of a real binary tensor of format $2 \times 2 \times \cdots \times 2$ (n times) are an n -tuple of quadratic invariants of the tensor. We introduce the *Gram locus*, the locus of tuples that arise as the Gram determinants of a real binary tensor. Here, the Gram locus is equal to the “set of feasible higher-order singular values”, from [8], under change of coordinates. The Gram determinants offer a convenient set of coordinates for studying the higher order singular values of a tensor.

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In [Theorem 1.3](#) we find the convex hull of the Gram locus for real binary tensors. It is a convex polytope that we describe explicitly. Its facet defining inequalities are that each Gram determinant is bounded by the sum of the others. We give a sum-of-squares proof. In [Theorem 1.4](#), we express the Gram locus as a semi-algebraic set for the case of $2 \times 2 \times 2$ tensors. The semi-algebraic description determines whether a tuple lies in the Gram locus or its complement, and characterizes tuples on the boundary. The non-linear part of the boundary of the Gram locus is [Fig. 4](#), and it is depicted in the highest higher order singular value coordinates in [Fig. 6](#).

In Section 1 of [\[8\]](#) it is conjectured that tensors with strictly decreasing and positive singular values in each flattening lie in the interior of the feasible set. [Example 3.1](#) is a counter-example to this conjecture. It is a tensor on the boundary of the feasible set whose higher order singular values in each flattening are distinct and positive. Its singular values are located at the black dot in [Fig. 6](#).

[Conjecture 1.5](#) proposes the general form for the Gram locus. It has a concise expression as the non-negativity of a single polynomial in the Gram determinants.

Finally, Section 4 gives a partial answer to [\[8, Problem 1.6\]](#), characterizing the tensors whose higher order singular values coincide. In the case of matrices agreement of singular values implies orthogonal equivalence, but this is not true for tensors. [Theorem 4.1](#) shows that the hyperdeterminant bridges the gap between orthogonal equivalence of tensors and agreement of the higher order singular value decomposition in the $2 \times 2 \times 2$ case. Our set-up for the $2 \times 2 \times 2$ tensor format is described in the following example.

Example 1.1. The $2 \times 2 \times 2$ tensor $(a_{ijk}), 0 \leq i, j, k \leq 1$, has eight entries which populate the vertices of the three-cube. It has three flattenings, each of size 2×4 :

$$\begin{bmatrix} a_{000} & a_{001} & a_{010} & a_{011} \\ a_{100} & a_{101} & a_{110} & a_{111} \end{bmatrix} \quad \begin{bmatrix} a_{000} & a_{001} & a_{100} & a_{101} \\ a_{010} & a_{011} & a_{110} & a_{111} \end{bmatrix} \quad \begin{bmatrix} a_{000} & a_{010} & a_{100} & a_{110} \\ a_{001} & a_{011} & a_{101} & a_{111} \end{bmatrix}.$$

For the i th flattening M we find $d_i := \det(MM^T)$. For instance, the first Gram determinant is

$$d_1 = (a_{000}a_{101} - a_{001}a_{100})^2 + (a_{000}a_{110} - a_{010}a_{100})^2 + (a_{000}a_{111} - a_{011}a_{100})^2 + (a_{001}a_{110} - a_{010}a_{101})^2 + (a_{001}a_{111} - a_{011}a_{101})^2 + (a_{010}a_{111} - a_{011}a_{110})^2.$$

A computation reveals that the linear combination $d_2 + d_3 - d_1$ can be written as a sum of three squared terms:

$$2(a_{000}a_{011} - a_{010}a_{001})^2 + 2(a_{100}a_{111} - a_{110}a_{101})^2 + (a_{010}a_{101} + a_{001}a_{110} - a_{011}a_{100} - a_{000}a_{111})^2.$$

The sum-of-squares certificate certifies that the expression is non-negative for all real values of the variables.

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