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On singularities of third secant varieties of Veronese embeddings ☆

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ABSTRACT

In this paper we study singularities of third secant varieties of Veronese embedding $v_d(\mathbb{P}^n)$, which corresponds to the variety of symmetric tensors of border rank at most three in $(\mathbb{C}^{n+1})^{\otimes d}$.

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1. Introduction

For a projective algebraic variety $X \subset \mathbb{P}W$, the *k-th secant variety* $\sigma_k(X)$ is defined by

$$\sigma_k(X) = \overline{\bigcup_{x_1 \cdots x_k \in X} \mathbb{P}\langle x_1 \cdots x_k \rangle} \subset \mathbb{P}W \tag{1.1}$$

where $\langle x_1 \cdots x_k \rangle \subset W$ denotes the linear span of the points $x_1 \cdots x_k$ and the overline denotes Zariski closure. Let V be an $(n+1)$-dimensional complex vector space and $W = S^d V$ be the subspace of symmetric d-way tensors in $V^{\otimes d}$. Equivalently, we can also think of W as the space of homogeneous polynomials of degree d in $n+1$ variables. When X is the Veronese embedding $v_d(\mathbb{P}V)$ of rank one symmetric d-way tensors over V in $\mathbb{P}W$, then $\sigma_k(X)$ is the variety of symmetric d-way tensors of border rank at most k (see Subsection 2.1 for terminology and details).

If X is an irreducible variety and $\sigma_k(X)$ its k-secant variety, then it is well known that

$$\mathrm{Sing}(\sigma_k(X)) \supseteq \sigma_{k-1}(X) \tag{1.2}$$

(e.g. see [3, Corollary 1.8]). Equality holds in many basic examples, like determinantal varieties defined by minors of a generic matrix, but the strict inequality also holds for some other tensors (e.g. just have a look at [12, Corollary 7.17] for the case $\sigma_2(X)$ when X is the Segre embedding $\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_r$ or [2, Fig. 1, p. 18] for the third secant variety of Grassmannian $\mathbb{G}(2,6)$).

Therefore, it should be very interesting to compute more cases and to give a general treatment about singularities of secant varieties. Further, the knowledge of singular locus is known to be very crucial to the so-called *identifiability problem*, which is to determine uniqueness of a tensor decomposition (see [6, Theorem 4.5]). It has recently been paid more attention in this context. In this paper, we deal with the case of third secant variety of Veronese embeddings, $\sigma_3(v_d(\mathbb{P}V))$.

From now on, let X be the Veronese variety $v_d(\mathbb{P}V)$ in $\mathbb{P}S^d V = \mathbb{P}^N$ with $N = \dim_{\mathbb{C}} S^d V - 1 = \binom{n+d}{n} - 1$. One could ask the following problem:

Problem 1.1. Let $V = \mathbb{C}^{n+1}$. Determine for which triple (k, d, n) it does hold that

$$\mathrm{Sing}(\sigma_k(v_d(\mathbb{P}V))) = \sigma_{k-1}(v_d(\mathbb{P}V))$$

for every $k \geq 2, d \geq 2$ and $n \geq 1$ or describe $\mathrm{Sing}(\sigma_k(v_d(\mathbb{P}V)))$ if it is not the case.

We'd like to remark here that our question is a set-theoretic one. First, it is classical that the answer to Problem 1.1 is true for the binary case (i.e. $n = 1$) (see e.g.