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Repetition invariant geometric means

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A B S T R A C T

We introduce a class of repetition invariant geometric means and obtain corresponding contractive barycentric maps of integrable Borel probability measures on the Cartan–Hadamard Riemannian manifold of positive definite matrices. They retain most of the properties of the Cartan barycenter and lead to the conclusion that there are infinitely many distinct contractive barycentric maps. Inequalities from the derived geometric means including the Yamazaki inequality and unitarily invariant norm inequalities are presented.

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1. Introduction

The Cartan mean (alternatively Riemannian mean) on the Riemannian manifold $\mathbb{P}$ of $m \times m$ positive definite Hermitian matrices equipped with the trace metric $\delta(A, B) = \| \log A^{-1/2} B A^{-1/2} \|_2$ has been extensively studied. It has recently become an important tool for the averaging and study of positive definite matrices. Many research topics about the Cartan mean such as finding properties and computing efficiently it have been launched. See [3–5,15,19,23] and references therein.

The Cartan mean of positive definite matrices $A_1, \ldots, A_n$ is the unique minimizer

$$\Lambda_n(A_1, \ldots, A_n) := \underset{X \in \mathbb{P}}{\arg\min} \sum_{j=1}^{n} \delta^2(A_j, X)$$

and is a multivariate extension of the two-variable geometric mean

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

Indeed, the curve $[0, 1] \ni t \mapsto A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}$ of weighted geometric means is the unique (up to parametrization) geodesic line from A to B, and hence, $A \# B = \Lambda_2(A, B)$ is the unique δ-midpoint between A and B. The Cartan mean arises also as a unique positive definite solution X of the Karcher equation (gradient zero equation)

$$\sum_{j=1}^{n} \log(X^{-1/2} A_j X^{-1/2}) = 0. \tag{1.1}$$

One of the important properties among others is the *repetition invariancy* from its defining equation

$$\Lambda_n(A_1, \ldots, A_n) = \Lambda_{nk}(\underline{A_1, \ldots, A_n}, \ldots, \underline{A_1, \ldots, A_n}),$$

where the number of blocks is k. This particular property has been less studied in detail in the theory of multivariate geometric means. However, it is a necessary condition for the extension of a mean to a barycentric map of Borel probability measures on $\mathbb{P}$. A metric space equipped with a contractive barycentric map with respect to the Wasserstein distance on the set of Borel probability measure with finite first moment is called a *barycentric metric space* and plays a fundamental role in the theory of integrations (random variables, expectations and variances), law of large numbers, Birkhoff ergodic theorem, Jensen's inequality [7,8,24,17,20], and optimal transport theory on Riemannian manifolds [21,22], to cite only a few. In [24] K.-T. Sturm developed a theory of barycenters of probability measures for metric spaces of nonpositive curvature, where the least squares barycenter is contractive for the Wasserstein metric, a property that has been called the fundamental contraction property.