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# Lie's correspondence for commutative automorphic formal loops

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ABSTRACT

We develop Lie's correspondence for commutative automorphic formal loops, which are natural candidates for non-associative abelian groups, to show how linearization techniques based on Hopf algebras can be applied to study non-linear structures. Over fields of characteristic zero, we prove that the category of commutative automorphic formal loops is equivalent to certain category of Lie triple systems. An explicit Baker–Campbell–Hausdorff formula for these loops is also obtained with the help of formal power series with coefficients in the algebra of 3 by 3 matrices. Our formula is strongly related to the function $\frac{(e^{2s}-e^{2t})(s+t)}{2(e^{2(s+t)}-1)}$.



## 1. Introduction

Loops are the non-associative counterpart of groups. These algebraic structures have a product $xy$ and a unit element $e$. Apart from this, the only extra requirement is that

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the left and right multiplication operators $L_x\colon y \mapsto xy$ and $R_x\colon y \mapsto yx$ are invertible for all $x$, which is equivalent to the existence of left and right divisions $x\backslash y$ and $x/y$ such that $x\backslash(xy) = y = x(x\backslash y)$ and $(yx)/x = y = (y/x)x$. The lack of associativity uncovers a tremendous rich 'phylogenetic tree' of varieties of loops that has motivated recent developments in non-associative mathematics. The present paper gives more evidence about the close relationship between commutative automorphic loops and abelian groups as 'non-associative species'.

The interest in loops began in the 1930s with the work of Moufang on projective geometry. Loops that satisfy $x(y(xz)) = ((xy)x)z$ are now called Moufang loops in her honor. In 1955 Malcev [24] noticed that Lie's approach to the study of local analytic groups might work even when associativity is relaxed. Under this new point of view Lie algebras are just the tangent algebras of associative analytic loops, but many other varieties of tangent algebras exist. Moufang analytic loops are diassociative–i.e., the subloop generated by any two elements is a group–and their tangent algebras, now called Malcev algebras, are binary-Lie algebras–i.e. the subalgebra generated by any two elements is a Lie algebra. Another interesting observation from Malcev was that the Baker–Campbell–Hausdorff formula only depends on two elements, thus the same formula makes sense for binary-Lie algebras. This suggested that finite-dimensional real Malcev algebras integrate to local analytic Moufang loops, as proved in 1970 by Kuzmin [21,22]. Since then, the study of Lie's correspondence in non-associative settings was a challenging problem (see for instance [2,4,8,12,18,31,34–36,38,39,42,46]), finally solved by Mikheev and Sabinin in 1987 with the apparatus of affine connections from differential geometry [40]. The tangent algebra of a local analytic loop is a Sabinin algebra–an algebraic structure with two infinite families of multilinear operations satisfying certain axioms–and, under certain convergence conditions, any finite-dimensional real Sabinin algebra is the tangent algebra of a (uniquely determined up to isomorphism) local analytic loop. See [45] for an excellent modern treatment.

While the result of Mikheev and Sabinin shows that Lie's correspondence remains valid even when associativity is removed, in practice it is difficult to compute the identities that define the varieties of Sabinin algebras associated to varieties of loops. For instance, the theory ensures the existence of two infinite families of multilinear operations on the tangent space of any local analytic Moufang loop that classify it; however, in practice, only a binary operation is required since the other multilinear operations can be derived from this one, and the axioms satisfied by this binary operation do not clearly follow from those of Sabinin algebra. Thus a case-by-case approach is required in the study of varieties of loops.

Over the years several varieties of loops and quasigroups have been studied in connection with geometry (see the books [3,5,7,9,37] and references therein) and new examples of Lie's correspondence have appeared. Recently, the variety of automorphic loops introduced in 1956 by Bruck and Paige [6] has attracted a lot of attention and it is an active area of research (see [44] for an updated account on the subject). These loops are defined by the following property: *the stabilizer of the unit element e in the group generated*