Linear Algebra and its Applications 543 (2018) 1–16

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The tensor rank of tensor product of two three-qubit W states is eight

Lin Chen [a,b,*], Shmuel Friedland [c]

[a] School of Mathematics and Systems Science, Beihang University, Beijing 100191, China

[b] International Research Institute for Multidisciplinary Science, Beihang University, Beijing 100191, China

[c] Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607-7045, USA

A R T I C L E   I N F O

Article history:
Received 29 November 2017
Accepted 15 December 2017
Available online 20 December 2017
Submitted by R. Brualdi

MSC:
15A69
15A72
46A32
46B28
46M05
47A80
53A45

Keywords:
Tensor rank
Qubit
W state
Kronecker product
Tensor product

A B S T R A C T

We show that the tensor rank of tensor product of two three-qubit W states is not less than eight. Combining this result with the recent result of M. Christandl, A.K. Jensen, and J. Zuiddam that the tensor rank of tensor product of two three-qubit W states is at most eight, we deduce that the tensor rank of tensor product of two three-qubit W states is eight. We also construct the upper bound of the tensor rank of tensor product of many three-qubit W states.

* Corresponding author.
E-mail addresses: linchen@buaa.edu.cn (L. Chen), friedlan@uic.edu (S. Friedland).

https://doi.org/10.1016/j.laa.2017.12.015

1. Introduction

Let $\mathbf{H}$ be an n-dimensional Hilbert space. We denote by a bold letter $\mathbf{x}$ an element in $\mathbf{H}$. For compactness of the exposition we adopt the following terminology. A nonzero vector $\mathbf{x}$ is called a state, while a normalized state is a vector $\mathbf{x}$ of norm one. For a positive integer $d > 1$ a d-partite state is the Hilbert space $\mathbf{H} = \mathbf{H}_1 \otimes \cdots \otimes \mathbf{H}_d$, where $\dim \mathbf{H}_i = n_i$ for $i \in [d] = \{1, \ldots, d\}$. We denote $\otimes_{i=1}^d \mathbf{H}_i = \mathbf{H}$. In the case $\mathbf{H}_1 = \ldots = \mathbf{H}_d$ we denote $\mathbf{H}$ by $\otimes^d \mathbf{H}_1$. An unentangled state is a rank one tensor $\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_d$, where $\mathbf{x}_i \neq \mathbf{0}, i \in [d]$. We denote by a calligraphic letter $\mathcal{X}$ an element of $\otimes_{i=1}^d \mathbf{H}_i$. The rank of a state $\mathcal{X}$, denoted by rank $\mathcal{X}$, is the minimal number r in the decomposition of $\mathcal{X}$ as a sum of unentangled states $\mathcal{X} = \sum_{j=1}^r \otimes_{i=1}^d \mathbf{x}_{i,j}$. Thus rank $\mathcal{X}$ is a measurement of entanglement of a state. There are other measure of entanglement of normalized states, as geometrical measure of entanglement [1,2] or the nuclear norm of $\mathcal{X}$ [3].

The entanglement of bipartite states, i.e. $d = 2$, is well understood, since $\mathbf{H}_1 \otimes \mathbf{H}_2$ can be identified with the space of $\dim \mathbf{H}_1 \times \dim \mathbf{H}_2$ matrices. In this case rank $\mathcal{X}$ is the rank of the corresponding matrix, and the maximal value of this rank is $\min(\dim \mathbf{H}_1, \dim \mathbf{H}_2)$. To emphasize that we are dealing with bipartite states, i.e. matrices, we will usually denote by X the matrix representing the bipartite state. The first interesting case is the 3-qubit states: $d = 3, \dim \mathbf{H}_1 = \dim \mathbf{H}_2 = \dim \mathbf{H}_3 = 2$. There are two kinds of entangled states which can not be decomposed as a product of an unentangled state with a two qubit entangled state: the GHZ and W states whose ranks are 2 and 3 respectively. The closure of the orbit of GHZ under the action of $GL(\mathbb{C}^2) \times GL(\mathbb{C}^2) \times GL(\mathbb{C}^2)$ is $\otimes^3 \mathbf{H}_1$, and its rank is two. The W state has the maximum rank three. We will usually denote the W state by the tensor $\mathcal{W}$.

We now consider another d' partite state Hilbert space $\mathbf{H}' = \otimes_{i'=1}^{d'} \mathbf{H}'_{i'}$, where $\dim \mathbf{H}'_{i'} = n'_{i'}, i' \in [d']$. We define two different tensor products of $\mathbf{H}$ and $\mathbf{H}'$. The first product is the tensor product $\mathbf{H} \otimes \mathbf{H}'$. It has the following physical interpretation. The d and d' partite tensor products $\mathbf{H}$ and $\mathbf{H}'$ correspond to two sets of parties $\{P_1, \ldots, P_d\}$ and $\{Q_1, \ldots, Q_{d'}\}$. Then $\mathbf{H} \otimes \mathbf{H}'$ corresponds to $d + d'$ party $\{P_1, \ldots, P_d, Q_1, \ldots, Q_{d'}\}$. The second tensor product, which we call the Kronecker product, is defined as follows. Assume that $d \leq d'$. (We can always achieve this by permuting the factors $\mathbf{H}$ and $\mathbf{H}'$.) Then

$$\mathbf{H} \otimes_K \mathbf{H}' = (\otimes_{i=1}^d (\mathbf{H}_i \otimes \mathbf{H}'_i)) \otimes (\otimes_{i'=d+1}^{d'} \mathbf{H}'_{i'}).$$

(If $d' = d$ the second tensor product is omitted.) The physical interpretation of the Kronecker product is as follows. The d and d' partite tensor products $\mathbf{H}$ and $\mathbf{H}'$ correspond to two sets of parties $\{P_1, \ldots, P_d\}$ and $\{P_1, \ldots, P_{d'}\}$ respectively. Then $\mathbf{H} \otimes_K \mathbf{H}'$ corresponds to the party $\{P_1, \ldots, P_{d'}\}$ where each person P_i has the space $\mathbf{H}_i \otimes \mathbf{H}'_i$ for $i \in [d]$. For $i' > d$ the person $P_{i'}$ has the space $\mathbf{H}'_{i'}$. Note that for $d = d' = 2$ $\mathbf{H} \otimes_K \mathbf{H}'$ corresponds to the Kronecker product two matrix spaces. Suppose that $\mathbf{H}' = \mathbf{H}$. Then $\otimes^p \mathbf{H} = \mathbf{H}^{\otimes p}$ is pd partite system corresponding to p tensor products of $\mathbf{H}$. Furthermore, $\otimes_K^p \mathbf{H} = \otimes_{i=1}^d (\otimes^p \mathbf{H}_i)$.