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Reduction of a pair of skew-symmetric matrices to its canonical form under congruence

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ABSTRACT

Let (A, B) be a pair of skew-symmetric matrices over a field of characteristic not 2. Its regularization decomposition is a direct sum

$$(\underline{A}, \underline{B}) \oplus (A_1, B_1) \oplus \cdots \oplus (A_t, B_t)$$

that is congruent to (A, B), in which $(\underline{A}, \underline{B})$ is a pair of nonsingular matrices and $(A_1, B_1), \ldots, (A_t, B_t)$ are singular indecomposable canonical pairs of skew-symmetric matrices under congruence. We give an algorithm that constructs a regularization decomposition. We also give a constructive proof of the known canonical form of (A, B) under congruence over an algebraically closed field of characteristic not 2.

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1. Introduction

We give an algorithm that for each pair of skew-symmetric matrices constructs its regularization decomposition.

Two pairs (A,B) and (A',B') of square matrices of the same size are *congruent* if there exists a nonsingular matrix S such that

$$S(A,B)S^T := (SAS^T, SBS^T) = (A',B').$$

A *direct sum* of pairs (A,B) and (A',B') is the pair

$$(A,B) \oplus (A',B') := \left(\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}, \begin{bmatrix} B & 0 \\ 0 & B' \end{bmatrix} \right).$$

A *regularizing decomposition* of a pair (A,B) of skew-symmetric matrices over a field of characteristic not 2 is a direct sum

$$(\underline{\underline{A}}, \underline{\underline{B}}) \oplus (A_1,B_1) \oplus \cdots \oplus (A_t,B_t) \tag{1}$$

that is congruent to (A,B), in which $(\underline{\underline{A}}, \underline{\underline{B}})$ is a pair of nonsingular matrices of the same size and each (A_i,B_i) is one of the pairs

$$\mathcal{J}_n := \left(\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix} \right), \tag{2}$$

$$\mathcal{K}_n := \left(\begin{bmatrix} 0 & J_n(0) \\ -J_n(0)^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right),$$

$$\mathcal{L}_n := \left(\begin{bmatrix} 0 & L_n \\ -L_n^T & 0 \end{bmatrix}, \begin{bmatrix} 0 & R_n \\ -R_n^T & 0 \end{bmatrix} \right), \quad n = 1, 2, \dots, \tag{3}$$

where $J_n(0)$ is the $n \times n$ singular Jordan block and

$$L_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \quad R_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix} \quad ((n-1)\text{-by-}n). \tag{4}$$

In particular, $\mathcal{L}_1 = ([0],[0])$. The canonical form of (A,B) under congruence (see (5)) ensures that $(\underline{\underline{A}}, \underline{\underline{B}})$—the *regular part* of (A,B)—is determined up to congruence, and $(A_1,B_1),\dots,(A_t,B_t)$—the *singular summands*—are determined uniquely up to permutations.

In Section 2, we give a *regularization algorithm* that uses elementary transformations of matrices and for each pair of skew-symmetric matrices over a field of characteristic not 2 constructs its regularization decomposition under congruence. Regularization algorithms were constructed for matrix pencils by Van Dooren [16], for cycles of linear