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A note on suborthogonal lattices

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A R T I C L E I N F O

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In this paper it is shown that, given any k-dimensional lattice Λ, there is a lattice sequence Λ_w, $w \in \mathbb{Z}$, with a suborthogonal lattice $\Lambda_o \subset \Lambda_w$, converging to Λ (up to equivalence). Also, we discuss conditions for the fastest convergence.

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1. Introduction

A large class of the problems in coding theory is related to the properties of the lattices, in special with sublattices generated by orthogonal basis (suborthogonal lattices). Several authors investigated the relationship of suborthogonal lattices with spherical codes, see [3,9,10,14], and with the q-ary codes, see [1,8,13,16–18], but, of course, this subject does not restrict to these problems [2,15,4,5,12].

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In general, the problems in lattices are concentrated in obtaining certain parameters such as: the shortest vector, the packing radius and the packing density, the coverage radius and the coverage density.

The points in lattices are interpreted as elements of a code. Thus finding efficient coding and decoding schemes is essential. There are several construction in the literature that establish the relationship of linear codes with lattices, a good reference is [7].

This article is organized as follows. In Section 2, we present the notations, definitions and small properties. In Section 3, we present a new scheme of obtaining lattices with a suborthogonal lattice. In Section 4, we present a case study for special lattices: $\mathbb{D}_n$ and $\mathbb{E}_n$ $(n = 7, 8)$ and Leech lattice Λ_{24}.

1.1. Background definitions and results

A *lattice* in $\mathbb{R}^n$ is a discrete additive subgroup of $\mathbb{R}^n$, Λ, which has a *generator matrix* with full rank, $k \times n$, $\mathcal{B}$, e.g, $\boldsymbol{v} \in \Lambda \leftrightarrow \boldsymbol{v} = \boldsymbol{u}^t\mathcal{B}$ ($\boldsymbol{u} \in \mathbb{Z}^k$, k is said rank of Λ. The *determinant* of a lattice is $\det(\Lambda) = \det(\mathcal{G})$, where $\mathcal{G} = \mathcal{B}\mathcal{B}^t$ is a *Gram matrix* of the lattice Λ and the *k-dimensional volume* of the lattice is $\sqrt{\det(\Lambda)}$ (volume of the parallelotope generated by the rows of $\mathcal{B}$). The *minimum norm* of the Lattice Λ, $\mu(\Lambda)$, is $\min\{\|\boldsymbol{v}\|; \boldsymbol{v} \in \Lambda \text{ and } \boldsymbol{v} \neq \boldsymbol{0}\}$ (Euclidean norm), the packing radius is $\rho(\Lambda) = \frac{1}{2}\mu(\Lambda)$ and the *center density packing* of Λ is $\delta_\Lambda = \dfrac{\rho(\Lambda)^k}{\mathrm{vol}(\Lambda)}$. Two lattices Λ_1 and Λ_2, with generator matrices $\mathcal{B}_1$ and $\mathcal{B}_2$ are *equivalent* if and only if $\mathcal{B}_1 = c\,\mathcal{U}\mathcal{B}_2\mathcal{O}$, where c is nonzero constant, $\mathcal{U}$ is an *unimodular matrix* ($k \times k$ integer matrix with $\det(\mathcal{U}) = \pm 1$) and $\mathcal{O}$ is a $n \times n$ real orthogonal matrix ($\mathcal{O}\mathcal{O}^t = \mathcal{I}_n$, $\mathcal{I}_n$ identity matrix $k \times k$). The *dual lattice* of Λ is a lattice, Λ^*, obtained for all vectors $\boldsymbol{u} \in \mathrm{span}(\mathcal{B})$ (there $\mathrm{span}(\mathcal{B})$ is a vector space generated by the rows of $\mathcal{B}$) such that $\boldsymbol{u} \cdot \boldsymbol{v} \in \mathbb{Z}, \forall \boldsymbol{v} \in \Lambda$. The generator matrix of Λ^* is $\mathcal{B}^* = (\mathcal{B}\mathcal{B}^t)^{-1}\mathcal{B}$, in particular, $\mathcal{B}^* = \mathcal{B}^{-t}$ if $n = k$. A sublattice, Λ', is a subset of Λ which is also a lattice and if Λ' has a generator matrix is formed by the orthogonal row vectors we will say that it is a *suborthogonal lattice*.

Since a lattice is a group, remember that the quotient of a lattice Λ by a sublattice Λ' (both of same rank k), $\frac{\Lambda}{\Lambda'}$, is as a finite abelian group with M elements, where M is the ratio of the volume of Λ' by the volume of Λ, i.e., $M = \frac{\mathrm{vol}(\Lambda')}{\mathrm{vol}(\Lambda)}$. The M elements of the lattice Λ can be seen as an orbit of a vector in the k-dimensional torus $\frac{\Lambda}{\Lambda'}$. This essentially establishes the relationship with central spherical class codes, as well as a class of the linear codes track construction “A” and similar constructions, see more details in [7].

2. Suborthogonal sequences

Consider a lattice $\Lambda \subset \mathbb{R}^n$ of rank n containing an orthogonal sublattice, $\Lambda_o \subset \Lambda$, such that Λ_o is equivalence to $\mathbb{Z}^n$, i.e., a generator matrix of Λ_o is $c\,\mathcal{O}$, with $\mathcal{O}\mathcal{O}^t = \mathcal{I}_n$. Let $\mathcal{B}$ and $\mathcal{B}^* = \mathcal{B}^{-t}$ be the generator matrices of Λ and Λ^* (respectively). Assuming that $\mathcal{B}^*$ has integer entries, then the lattice, Λ, with generator matrix $\mathcal{B} = \mathrm{adj}(\mathcal{B}^*) = \det(\mathcal{B}^*)\mathcal{B}^{*-t}$,

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