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A complete characterization of determinantal quadratic polynomials

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ABSTRACT

The problem of expressing a multivariate polynomial as the determinant of a monic (definite) symmetric or Hermitian linear matrix polynomial (LMP) has drawn a huge amount of attention due to its connection with optimization problems. In this paper we provide a necessary and sufficient condition for the existence of *monic Hermitian determinantal representation* as well as *monic symmetric determinantal representation* of size 2 for a given quadratic polynomial. Further we propose a method to construct such a monic determinantal representation (MDR) of size 2 if it exists. It is known that a quadratic polynomial $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + b^T\mathbf{x} + 1$ has a symmetric MDR of size $n+1$ if A is *negative semidefinite*. We prove that if a quadratic polynomial $f(\mathbf{x})$ with A which is not negative semidefinite has an MDR of size greater than 2, then it has an MDR of size 2 too. Finally, we characterize all quadratic polynomials that exhibit MDRs of any size.

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1. Introduction

This paper deals with characterization of quadratic real multivariate polynomials which admit monic Hermitian (symmetric) determinantal representations, that is polynomials which can be written as the determinant of a monic linear matrix polynomial (LMP) whose coefficient matrices are Hermitian (symmetric) and the constant coefficient matrix is the identity matrix. Note that the coefficient matrices of the LMP could be of any size greater than one. In particular, in this paper, we focus on the existence and determination of a monic LMP whose coefficient matrices are Hermitian (symmetric) and of size 2 for a given quadratic polynomial. Besides, we identify the class of quadratic polynomials for which an MDR of size greater than 2 ensures the existence of an MDR of size 2.

Determinantal representations of polynomials have generated a lot of interest due to its connection with the problem of determining (definite) LMI representable sets, also known as spectrahedra [1] which play a crucial role in optimization problems. Indeed, if the feasible set of an optimization problem is a definite LMI representable set, the problem can be transformed into a semidefinite programming (SDP) problem which in turn can be efficiently solved by SDP solvers. It is important to recall that any polynomial can be expressed as the determinant of a symmetric LMP [10]. It is also known that the algebraic interior defined by a real zero (RZ) quadratic polynomial is always a spectrahedron, since a Hermitian determinantal representation can be obtained for higher powers of RZ quadratic polynomials using Clifford algebra [8] and sum of squares (SOS) decomposition of a parametrized Hermite matrix [7]. To the best of authors' knowledge, existence and characterization of quadratic polynomials that have an MDR of size 2 has not been done before.

In this paper, we provide a necessary and sufficient condition for the existence of MDRs of size 2 for a quadratic polynomial. We also propose a constructive method to determine these MDRs. We show that for a certain sub-class of quadratic polynomials that have MDRs of size 2, there are precisely two non-equivalent classes of MDRs, whereas for all other quadratic polynomials that have MDRs of size 2, all MDRs are unitarily equivalent. Recall that a quadratic polynomial $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + b^T\mathbf{x} + 1$ admits a symmetric MDR of size $n + 1$ if the corresponding matrix A is negative semidefinite [11], [3]. This need not imply the existence of MDR of size 2 for the same polynomial. Therefore, it is natural to ask whether quadratic polynomials which have an MDR of size 2 can be characterized. We further characterize all quadratic polynomials that have an MDR of any size [Sec 4]. We show that if a quadratic polynomial $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} + b^T\mathbf{x} + 1$ has an MDR, then either A is negative semidefinite or $f(\mathbf{x})$ admits an MDR of size 2. In other words, if $f(\mathbf{x})$ has an MDR, but A is not negative semidefinite, then $f(\mathbf{x})$ has an MDR of size 2.

2. Preliminaries

We begin with the concept of definite LMI representable sets and its relation to monic determinantal representations. A set $S \subseteq \mathbb{R}^n$ is said to be *LMI representable* if