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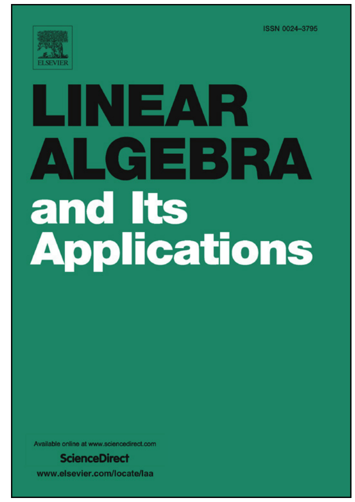

Tensor rank is not multiplicative under the tensor product

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Abstract

The tensor rank of a tensor t is the smallest number r such that t can be decomposed as a sum of r simple tensors. Let s be a k-tensor and let t be an ℓ-tensor. The tensor product of s and t is a $(k+\ell)$-tensor. Tensor rank is sub-multiplicative under the tensor product. We revisit the connection between restrictions and degenerations. A result of our study is that tensor rank is not in general multiplicative under the tensor product. This answers a question of Draisma and Saptharishi. Specifically, if a tensor t has border rank strictly smaller than its rank, then the tensor rank of t is not multiplicative under taking a sufficiently hight tensor product power. The "tensor Kronecker product" from algebraic complexity theory is related to our tensor product but different, namely it multiplies two k-tensors to get a k-tensor. Nonmultiplicativity of the tensor Kronecker product has been known since the work of Strassen.

It remains an open question whether border rank and asymptotic rank are multiplicative under the tensor product. Interestingly, lower bounds on border rank obtained from generalised flattenings (including Young flattenings) multiply under the tensor product.

Keywords: tensor rank, border rank, degeneration, Young flattening, algebraic complexity theory, quantum information theory
2010 MSC: 15A69

1. Introduction

Let U_i, V_i be finite-dimensional vector spaces over a field $\mathbb{F}$. Let t be a k-tensor in $U_1 \otimes \cdots \otimes U_k$. The *tensor rank* of t is the smallest number r such that t can be written as a sum of r simple tensors $u_1 \otimes \cdots \otimes u_k$ in $U_1 \otimes \cdots \otimes U_k$, and is denoted by $\mathrm{R}(t)$. Letting $\mathbb{F}$ be the complex numbers $\mathbb{C}$, the *border rank* of t is the smallest number r such that t is a limit point (in the Euclidean topology) of a sequence of tensors in $U_1 \otimes \cdots \otimes U_k$ of rank at most r, and is denoted by $\underline{\mathrm{R}}(t)$.

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