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A resolution of Paz's conjecture in the presence of a nondegatory matrix

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ABSTRACT

Let $M_n(\mathbb{F})$ be the algebra of $n \times n$ matrices over the field $\mathbb{F}$ and let $\mathcal{S}$ be a generating set of $M_n(\mathbb{F})$ as an $\mathbb{F}$-algebra. The length of a finite generating set $\mathcal{S}$ of $M_n(\mathbb{F})$ is the smallest number k such that words of length not greater than k generate $M_n(\mathbb{F})$ as a vector space. It is a long standing conjecture of Paz that the length of any generating set of $M_n(\mathbb{F})$ cannot exceed $2n-2$. We prove this conjecture under the assumption that the generating set $\mathcal{S}$ contains a nonderogatory matrix. In addition, we find linear bounds for the length of generating sets that include a matrix with some conditions on its Jordan canonical form. Finally, we investigate cases when the length $2n-2$ is achieved.

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1. Introduction

Let $\mathcal{A}$ be an associative finite-dimensional algebra over an arbitrary field $\mathbb{F}$ and let $\mathcal{S} = \{a_1, \dots, a_k\}$ denote a finite generating system of this algebra. We define the length function of a generating set and of the algebra as follows.

Definition 1.1. A *word* in $\mathcal{S}$ is a product of elements from $\mathcal{S}$. The *length* of the word $a_{i_1} \dots a_{i_t}$ where $a_{i_j} \in \mathcal{S}$ is equal to t. Furthermore, if A is a unitary algebra, we define 1 to be a word *of length* 0 (the empty word).

For $i \geq 0$ we denote $\mathcal{S}^i$ to be the set of all words of length not greater than i over $\mathcal{S}$, and $\mathcal{L}_i(\mathcal{S}) = \langle \mathcal{S}^i \rangle$, where $\langle \mathcal{X} \rangle$ denotes the linear span of a set $\mathcal{X}$ in a vector space over $\mathbb{F}$. Note that $\mathcal{L}_0(\mathcal{S}) = \langle 1_{\mathcal{A}} \rangle = \mathbb{F}$ for unitary algebras, and $\mathcal{L}_0(\mathcal{S}) = 0$, otherwise. Let

$$\mathcal{L}(\mathcal{S}) = \bigcup_{i=0}^{\infty} \mathcal{L}_i(\mathcal{S})$$

be the linear span of all words in the alphabet $\mathcal{S}$.

Definition 1.2. A word w of length l is said to be *reducible* if $w \in \mathcal{L}_i(\mathcal{S})$ for some $i < l$.

Definition 1.3. The *length of a generating system* $\mathcal{S}$ for the finite-dimensional algebra $\mathcal{A}$ is the number $l(\mathcal{S}) = \min\{k \in \mathbb{Z}_+ : \mathcal{L}_k(\mathcal{S}) = \mathcal{A}\}$, and the *length of the algebra* $\mathcal{A}$ is defined to be the number $l(\mathcal{A}) = \max\{l(\mathcal{S}) : \mathcal{L}(\mathcal{S}) = \mathcal{A}\}$.

Denote by $M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over the field $\mathbb{F}$, and denote by $M_{n,m}(\mathbb{F})$ the space of $n \times m$ matrices over $\mathbb{F}$. We define the following notation for some special matrices from $M_n(\mathbb{F})$:

- By E_{ij} we denote (i, j)-th matrix unit, that is, the matrix with 1 in (i, j)-th position and zeros elsewhere. (We do not specify the size of the matrix, as it will be clear from the context.)
- By I_n and O_n we denote the identity matrix and the zero matrix in $M_n(\mathbb{F})$.
- For any $\lambda \in \mathbb{F}$ we set $J_n(\lambda) = \lambda I_n + \sum_{i=1}^{n-1} E_{i,i+1} \in M_n(\mathbb{F})$, that is, the Jordan block of size n corresponding to the eigenvalue λ, and define $J_n = J_n(0)$.

If the size of the matrix is clear, we denote the aforementioned matrices as I, O and J, correspondingly.

Furthermore, e_i, $i = 1, \dots, n$, will denote the i-th vector of the standard basis of $\mathbb{F}^n$ over $\mathbb{F}$, that is, the column vector with n coordinates such that there is 1 in the i-th position and zeros elsewhere.

The problem of length computation for the matrix algebra $M_n(\mathbb{F})$ as a function of the size of matrices was posed in [13] and is still open. The only known upper bounds are due to Paz and Pappacena: