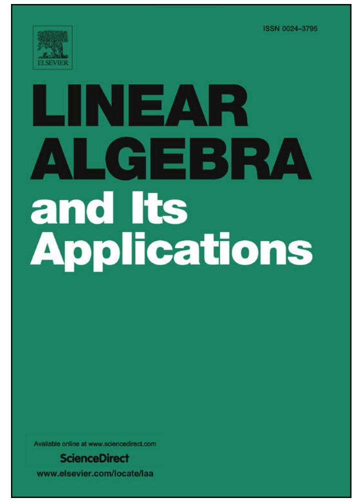
Accepted Manuscript

The relative operator entropy and the Karcher mean

Jun Ichi Fujii, Yuki Seo



PII: S0024-3795(16)30544-4
DOI: http://dx.doi.org/10.1016/j.laa.2016.11.018
Reference: LAA 13935

To appear in: *Linear Algebra and its Applications*

Received date: 26 July 2016
Accepted date: 10 November 2016

Please cite this article in press as: J.I. Fujii, Y. Seo, The relative operator entropy and the Karcher mean, *Linear Algebra Appl.* (2017), http://dx.doi.org/10.1016/j.laa.2016.11.018

The relative operator entropy and the Karcher mean

Jun Ichi Fujii and Yuki Seo

Abstract

The Karcher mean for positive invertible operators on a Hilbert space, which is the unique solution of the Karcher operator equation, was established by Lawson-Lim based on the geometric considerations for positive invertible operators. In this note, we extend both this mean and the equation to the non-invertible case. The key concept is the relative operator entropy. For operators whose ranges are closed, we show that the Karcher mean for the non-invertible case is the unique solution of the extended Karcher equation. As a byproduct, we can show the uniqueness of the Karcher solution for the invertible case based only on that of the power means.

1 Introduction

The Umegaki relative entropy $s_U(A|B) = \mathrm{Tr}A(\log A - \log B)$ [27] was introduced in the context of semi-finite von Neumann algebras as an extension of the Kullback-Leibler divergence. Moreover Uhlmann [26] extended it to quadratic interpolations of the corresponding geometric means of semi-norms by virtue of $\inf_{t>0} \dfrac{x^t - 1}{t} = \log t$. Of course, these entropies sometimes diverge, so that a certain condition is required to exist as a finite quantity.

On the other hand, derived from the theory of means of Pusz-Woronowicz [23, 24], Kubo and Ando [19] established the theory of operator means for positive operators on a Hilbert space (see also [3]):

$$A \,\mathrm{m}\, B = A^{\frac{1}{2}} f_{\mathrm{m}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} \qquad \text{for} \quad f_{\mathrm{m}}(x) = 1 \,\mathrm{m}\, x$$

where f_{m} is an operator monotone function and, by the monotonicity of each terms, $A \,\mathrm{m}\, B = \text{s-}\lim_{\varepsilon \to 0}(A+\varepsilon)\mathrm{m}(B+\varepsilon)$ defines an operator mean for all positive operators. Based on this theory, in [14] we introduced the relative operator entropy $\mathsf{S}(A|B)$ putting $f_{\mathrm{m}}(x) = \log x$, which is a relative version of the operator entropy defined by Nakamura-Umegaki [22]. From the viewpoint of Uhlmann, it also defined as the derivative at $t = 0$ of the path of geometric operator means [15] for $f_{\mathrm{m}_t}(x) = x^t$ $(t \in [0,1])$;

$$A \#_t B = \underset{\varepsilon \downarrow 0}{\text{s-lim}} (A+\varepsilon)^{\frac{1}{2}} \left((A+\varepsilon)^{-\frac{1}{2}} B (A+\varepsilon)^{-\frac{1}{2}} \right)^t (A+\varepsilon)^{\frac{1}{2}}.$$

For invertible A and B, the relative operator entropy has the following variational forms;

$$\mathsf{S}(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} = \lim_{t \downarrow 0} \frac{A \#_t B - A}{t}.$$

2010 *Subject Classification.* Primary 47A64; Secondary 47A63, 47A58, 94A17

Key words and phrases. Operator mean, relative operator entropy, Karcher equation, Karcher mean, power mean