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Abstract

The Karcher mean for positive invertible operators on a Hilbert space, which is the unique solution of the Karcher operator equation, was established by Lawson-Lim based on the geometric considerations for positive invertible operators. In this note, we extend both this mean and the equation to the non-invertible case. The key concept is the relative operator entropy. For operators whose ranges are closed, we show that the Karcher mean for the non-invertible case is the unique solution of the extended Karcher equation. As a byproduct, we can show the uniqueness of the Karcher solution for the invertible case based only on that of the power means.

1 Introduction

The Umegaki relative entropy $s_U(A|B) = \text{Tr}A(\log A - \log B)$ [27] was introduced in the context of semi-finite von Neumann algebras as an extension of the Kullback-Leibler divergence. Moreover Uhlmann [26] extended it to quadratic interpolations of the corresponding geometric means of semi-norms by virtue of $\inf_{t>0} \frac{x^t - 1}{t} = \log x$. Of course, these entropies sometimes diverge, so that a certain condition is required to exist as a finite quantity.

On the other hand, derived from the theory of means of Pusz-Woronowicz [23, 24], Kubo and Ando [19] established the theory of operator means for positive operators on a Hilbert space (see also [3]):

$$A \#_m B = A^{\frac{1}{2}} f_m \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} \quad \text{for } f_m(x) = 1 \#_m x$$

where f_m is an operator monotone function and, by the monotonicity of each terms, $A \#_m B = \text{s-lim}_{\varepsilon \rightarrow 0} (A + \varepsilon) \#_m (B + \varepsilon)$ defines an operator mean for all positive operators. Based on this theory, in [14] we introduced the relative operator entropy $S(A|B)$ putting $f_m(x) = \log x$, which is a relative version of the operator entropy defined by Nakamura-Umegaki [22]. From the viewpoint of Uhlmann, it also defined as the derivative at $t = 0$ of the path of geometric operator means [15] for $f_{m_t}(x) = x^t$ ($t \in [0, 1]$);

$$A \#_t B = \text{s-lim}_{\varepsilon \downarrow 0} (A + \varepsilon)^{\frac{1}{2}} \left((A + \varepsilon)^{-\frac{1}{2}} B (A + \varepsilon)^{-\frac{1}{2}} \right)^t (A + \varepsilon)^{\frac{1}{2}}.$$

For invertible A and B , the relative operator entropy has the following variational forms;

$$S(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right) A^{\frac{1}{2}} = \lim_{t \downarrow 0} \frac{A \#_t B - A}{t}.$$

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