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Edge-minimal graphs of exponent 2

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A R T I C L E I N F O

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A B S T R A C T

A simple undirected graph G has the *me*$_2$*-property* if every pair of distinct vertices of G is connected by a path of length 2, but this property does not survive the deletion of an edge. This article considers graphs that can be embedded as induced subgraphs of me$_2$-graphs by the introduction of additional mutually non-adjacent vertices and suitably chosen edges. The main results concern such embeddings of trees.

1. Introduction

A graph G is *primitive* if there exists a positive integer k with the property that if u and v are any vertices of G, then there exists a walk of length k from u to v in G. The least positive integer k with this property is called the *exponent* of the primitive graph. In this article we consider graphs that have exponent 2 and are *edge-minimal* with this property, in the sense that the deletion of any edge would result in a graph not having

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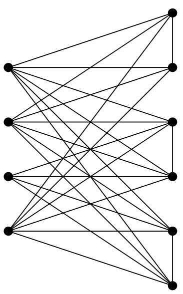
exponent 2. We will describe such graphs as me_2 (minimal exponent 2)-graphs and say that they have the *me_2-property*.

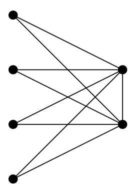
In this article we consider only simple undirected graphs, although the concepts of primitivity and exponent may be applied to much more general classes of graphs. A non-negative matrix A is *primitive* if A^k is positive for some integer k; the least such k is the *exponent* of A. Thus a graph is primitive if and only if its adjacency matrix is a primitive matrix, and in this case the exponents of the graph and its adjacency matrix coincide. General background on primitive graphs and matrices can be found in [1].

A simple graph G has the me_2-property if and only if G has exponent 2 and every edge of G is part of the *unique* path of length 2 between some pair of vertices. Since every pair of adjacent vertices in an me_2-graph must be connected by a path of length 2, every edge must belong to at least one triangle. Examples of me_2-graphs include the well-known *windmill* graphs, which are the only finite graphs with the property that every pair of vertices share exactly one neighbour (this statement is the Erdös–Rényi–Sós Theorem or Friendship Theorem, see [2]).

If uv and vw are edges of G (with $u \neq w$), we will say that uvw is a *unique 2-path* in G if v is the only mutual neighbour of u and w. The me_2-property may be viewed as a relaxation of the friendship property, which requires a unique 2-path between *every* pair of distinct vertices. The me_2-property allows the possibility of multiple 2-paths between some pairs of distinct vertices, but requires unique 2-paths to be sufficiently plentiful that every edge belongs to one. One family of me_2-graphs arises from taking unions of isomorphic windmills as follows. We write W_r for the "windmill with r blades", which has $2r+1$ vertices and consists of r triangles all having a single vertex in common and being otherwise disjoint. For positive integers k and r, let $W_{k,r}$ be the graph on $k+2r$ vertices obtained by taking k copies of W_r and identifying the tips of their blades together. For example, the diagram below shows the graphs $W_{4,3}$ and $W_{4,1}$.



$W_{4,3}$



$W_{4,1}$

It is easily verified that these graphs have exponent 2, that there is a unique 2-path from each of the k vertices on the left to each of the $2r$ vertices on the right, and hence that every edge of the graph belongs to a unique 2-path.

In [4], Kim et al. show that the minimum possible number of edges in a simple graph of exponent 2 and order n is $\frac{1}{2}(3n-3)$ if n is odd and $\frac{1}{2}(3n-2)$ if n is even. Graphs

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