ARTICLE IN PRESS LAA:14041

Linear Algebra and its Applications ••• (••••) •••–•••

Contents lists available at ScienceDirect

Linear Algebra and its Applications

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On the subdifferential of symmetric convex functions of the spectrum for symmetric and orthogonally decomposable tensors

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A R T I C L E I N F O

Article history:
Received 10 July 2016
Accepted 2 February 2017
Available online xxxx
Submitted by M. Van Barel

MSC:
15A69

Keywords:
Tensor
Schatten norm
Spectrum
Subdifferential

A B S T R A C T

The subdifferential of convex functions of the singular spectrum of real matrices has been widely studied in matrix analysis, optimization and automatic control theory. Convex optimization over spaces of tensors is now gaining much interest due to its potential applications in signal processing, statistics and engineering. The goal of this paper is to present an extension of the approach by Lewis [16] for the analysis of the subdifferential of certain convex functions of the spectrum of symmetric tensors. We give a complete characterization of the subdifferential of Schatten-type tensor norms for symmetric tensors. Some partial results in this direction are also given for Orthogonally Decomposable tensors.

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http://dx.doi.org/10.1016/j.laa.2017.02.003

1. Introduction

1.1. Background

Multidimensional arrays, also known as tensors are higher-order generalizations of vectors and matrices. In recent years, they have been the subject of extensive interest in various extremely active fields such as e.g. statistics, signal processing, automatic control, etc. ... where a lot of problems involve quantities that are intrinsically multidimensional such as higher order moment tensors [2]. Many natural and useful quantities in linear algebra such as the rank or the Singular Value Decomposition turn out to be very difficult to compute or generalize in the tensor setting [12,13,9]. Fortunately, efficient approaches exist in the case of symmetric tensors which lie at the heart of the moment approach which recently proved to be very efficient for addressing essential problems in Statistics/Machine Learning such as Clustering, estimation in Hidden Markov Chains, etc. .... See the very influential paper [2] for more details. In many statistical models such as the ones presented in [2], the rank of the involved is low and one expects that the theory of sparse recovery can be applied to recover them from just a few observations just as in the case of Matrix Completion [4,5], Robust PCA [3] and Matrix Compressed Sensing [18]. In such approaches to Machine Learning, one usually has to solve a penalized least squares problem of the type

$$\min_{X \in \mathbb{R}^{n_1 \times n_2}} \quad \|y - \mathcal{A}(X)\| + \lambda \; p(X),$$

where the penalization p is rank-sparsity promoting such as the nuclear norm and $\mathcal{A}$ is a linear operator taking values in $\mathbb{R}^n$. In the tensor setting, we look for solutions of problems of the type

$$\min_{X \in \mathbb{R}^{n_1 \times \cdots \times n_D}} \quad \|y - \mathcal{A}(X)\| + \lambda \; p(X),$$

for $D > 2$ and p being a generalization of the nuclear norm or some Schatten-type norm for tensors. The extension of Schatten norms to the tensor setting has to be carefully defined. In particular, several nuclear norms can be naturally defined [21,8,17]. Moreover, the study of the efficiency of sparsity promoting penalization relies crucially on the knowledge of the subdifferential of the norm involved as achieved in [1] or [15], or at least a good approximation of this subdifferential [21,17]. In the matrix setting, the works of [20,16] are famous for providing a neat characterization of the subdifferential of matrix norms or more generally functions of the matrix enjoying enough symmetries. In the 3D or higher dimensional setting, however, the case is much less understood. The relationship between the tensor norms and the norms of the flattenings is intricate although some good bounds relating one to the other can be obtained as in [11]. Notice that many recent works use the nuclear norms of the flattenings of the tensors to be optimized, especially in the field of compressed sensing; see e.g. [17,8]. One noticeable

Please cite this article in press as: S. Chrétien, T. Wei, On the subdifferential of symmetric convex functions of the spectrum for symmetric and orthogonally decomposable tensors, Linear Algebra Appl. (2017), http://dx.doi.org/10.1016/j.laa.2017.02.003