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Numerical investigation of Crouzeix's conjecture

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ABSTRACT

Crouzeix's conjecture states that for all polynomials p and matrices A, the inequality $\|p(A)\| \leq 2\,\|p\|_{W(A)}$ holds, where the quantity on the left is the 2-norm of the matrix $p(A)$ and the norm on the right is the maximum modulus of the polynomial p on $W(A)$, the field of values of A. We report on some extensive numerical experiments investigating the conjecture via nonsmooth minimization of the Crouzeix ratio $f \equiv \|p\|_{W(A)}/\|p(A)\|$, using Chebfun to evaluate this quantity accurately and efficiently and the BFGS method to search for its minimal value, which is 0.5 if Crouzeix's conjecture is true. Almost all of our optimization searches deliver final polynomial-matrix pairs that are very close to nonsmooth stationary points of f with stationary value 0.5 (for which $W(A)$ is a disk) or smooth stationary points of f with stationary value 1 (for which $W(A)$ has a corner). Our observations have led us to some additional conjectures as well as some new theorems. We hope that these give insight into Crouzeix's conjecture, which is strongly supported by our results.

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1. Crouzeix's conjecture

Let $\mathcal{M}^n$ denote the space of $n \times n$ complex matrices, let $\mathcal{P}_m$ denote the space of polynomials with complex coefficients and degree $\leq m$, and let $\|\cdot\|$ denote the 2-norm. For $A \in \mathcal{M}^n$, the field of values (or numerical range) of A is

$$W(A) = \{v^* A v : v \in \mathbb{C}^n, \|v\| = 1\} \subset \mathbb{C}.$$

The Toeplitz–Hausdorff theorem states that $W(A)$ is convex for all $A \in \mathcal{M}^n$ [18, Ch 1].

Let $p \in \mathcal{P}_m$ and let $A \in \mathcal{M}^n$. In 2004, M. Crouzeix conjectured [8] that for all m and n,

$$\|p(A)\| \leq 2\, \|p\|_{W(A)}. \tag{1}$$

The left-hand side is the 2-norm (spectral norm, maximum singular value) of the matrix $p(A)$, while $\|p\|_{W(A)}$ on the right-hand side is $\max_{\zeta \in W(A)} |p(\zeta)|$. By the maximum modulus theorem, $\|p\|_{W(A)}$ must be attained on bd $W(A)$, the boundary of $W(A)$.

In 2007, Crouzeix proved [9] that

$$\|p(A)\| \leq 11.08\, \|p\|_{W(A)} \tag{2}$$

i.e., the conjecture is true if we replace 2 by 11.08. Crouzeix wrote:

> The estimate 11.08 is not optimal. There is no doubt that refinements are possible which would decrease this bound. We are convinced that our estimate is very pessimistic, but to improve it drastically (recall that our conjecture is that 11.08 can be replaced by 2), it is clear that we have to find a completely different method.

The example

$$p(\zeta) = \zeta - \lambda,\ A = \begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix},$$

where $\alpha, \lambda \in \mathbb{C}$, $\alpha \neq 0$, shows that 11.08 cannot be replaced by a *smaller* number than 2. In this case, $W(A)$ is the disk of radius $|\alpha|/2$ centered at λ so $\|p\|_{W(A)} = |\alpha|/2$, and $\|p(A)\| = |\alpha|$.

As the degree of p is unbounded in Crouzeix's conjecture (1) and theorem (2), they can be extended from polynomials to functions analytic in the interior of $W(A)$ and continuous on its boundary. This is because $W(A)$ is a compact subset of the complex plane such that $\mathbb{C} \backslash W(A)$ is connected, and by Mergelyan's theorem [24,25] any function analytic on the interior of such a set and continuous on its boundary can be uniformly approximated by polynomials. The conjecture and theorem can also be extended from matrix space to infinite-dimensional Hilbert space, where the only difference is that $W(A)$