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A quadratically convergent algorithm based on matrix equations for inverse eigenvalue problems

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ABSTRACT

We propose a quadratically convergent algorithm for inverse symmetric eigenvalue problems based on matrix equations. The basic idea is seen in a recent study by Ogita and Aishima, while they derive an efficient iterative refinement algorithm for symmetric eigenvalue problems using special matrix equations. In other words, this study is interpreted as a unified view on quadratically convergent algorithms for eigenvalue problems and inverse eigenvalue problems based on matrix equations. To the best of our knowledge, such a unified development of algorithms is provided for the first time. Since the proposed algorithm for the inverse eigenvalue problems can be regarded as the Newton's method for the matrix equations, the quadratic convergence is naturally proved. Our algorithm is interpreted as an improved version of the Cayley transform method for the inverse eigenvalue problems. Although the Cayley transform method is one of the effective iterative methods, the Cayley transform takes $\mathcal{O}(n^3)$ arithmetic operations to produce an orthogonal matrix using a skew-symmetric matrix in each iteration. Our algorithm can refine orthogonality without the Cayley transform, which reduces the operations in each iteration. It is worth noting that our approach overcomes the limitation of the Cayley transform method to the inverse standard

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eigenvalue problems, resulting in an extension to inverse generalized eigenvalue problems.

1. Introduction

Let $A_0, A_1, \ldots, A_n$ be real symmetric $n \times n$ matrices and $\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*$ be real numbers. In addition, let $\boldsymbol{c} = [c_1, \ldots, c_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\Lambda^* = \mathrm{diag}(\lambda_1^*, \ldots, \lambda_n^*)$. Define

$$A(\boldsymbol{c}) := A_0 + c_1 A_1 + \cdots + c_n A_n \tag{1}$$

and denote its eigenvalues by $\lambda_1(\boldsymbol{c}) \leq \lambda_2(\boldsymbol{c}) \leq \cdots \leq \lambda_n(\boldsymbol{c})$ in the ascending order. A typical inverse eigenvalue problem is to find a solution $\boldsymbol{c}^* \in \mathbb{R}^n$ such that $\lambda_i(\boldsymbol{c}^*) = \lambda_i^*$ for all $1 \leq i \leq n$. Such inverse eigenvalue problems often arise in inverse vibration problems, inverse Sturm–Liouville problems, and nuclear spectroscopy [7,8,14]. In this study, we focus on numerical algorithms for solving the above inverse eigenvalue problems. As in [1,3,4,14], we assume that the prescribed eigenvalues are all distinct, i.e.,

$$\lambda_1^* < \lambda_2^* < \cdots < \lambda_n^*. \tag{2}$$

Let $X^* \in \mathbb{R}^{n \times n}$ denote an orthogonal matrix whose columns are the eigenvectors of $A(\boldsymbol{c}^*)$. Throughout the paper, I is an identity matrix and O is a zero matrix. For any matrix, let $\|\cdot\|$ denote the spectral norm and $[\cdot]_{ij}$ denote the (i, j) elements for $1 \leq i, j \leq n$.

In this paper, we propose a new iterative algorithm to solve the inverse eigenvalue problems. The basic idea to design the proposed algorithm is seen in a recent study by Ogita–Aishima [17], while they propose an efficient iterative refinement algorithm for symmetric eigenvalue problems. Similarly to [17], our algorithm for the inverse eigenvalue problems is derived as follows. For computed matrices $X^{(k)} \in \mathbb{R}^{n \times n}$ $(k = 0, 1, \ldots)$ in the iterative process, define $E^{(k)} \in \mathbb{R}^{n \times n}$ $(k = 0, 1, \ldots)$ such that $X^{(k)} = X^*(I + E^{(k)})$. Then we compute $\widetilde{E}^{(k)}$ approximating $E^{(k)}$ from the following relations:

$$\begin{cases} X^{*\mathrm{T}} X^* = I \\ X^{*\mathrm{T}} A(\boldsymbol{c}^*) X^* = \Lambda^* \end{cases}. \tag{3}$$

After computing $\widetilde{E}^{(k)}$, we can update $X^{(k+1)} := X^{(k)}(I - \widetilde{E}^{(k)})$, where $I - \widetilde{E}^{(k)}$ is the first order approximation of $(I + \widetilde{E}^{(k)})^{-1}$ using the Neumann series. Under some conditions, we prove $E^{(k)} \to O$ and $X^{(k)} \to X^*$ as $k \to \infty$, where the convergence rates are quadratic. Our algorithm can be regarded as the Newton's method for the matrix equations corresponding to (3).

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