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Unitary similarity of the determinantal representation of unitary bordering matrices

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ABSTRACT

Let $F(x, y, z)$ be a hyperbolic ternary form of degree n. The Helton–Vinnikov theorem asserts that there exists an $n \times n$ complex symmetric matrix S such that $F_S(x, y, z) = F(x, y, z)$, where the determinantal hyperbolic ternary form of an $n \times n$ matrix B is defined by $F_B(x, y, z) = \det(x\,\Re(B) + y\,\Im(B) + z\,I_n)$. Let A be an $n \times n$ unitary bordering matrix. It is known that A is unitarily similar to a symmetric matrix. In this paper, we investigate the unitary similarity between the unitary bordering matrix A and the Helton–Vinnikov symmetric matrix S admitted the determinantal representation of the ternary form $F_A(x, y, z)$ satisfying $F_S(x, y, z) = F_A(x, y, z)$ for $n = 3, 4$.

1. Introduction

Let A be an $n \times n$ complex matrix. The numerical range of A is defined as the set

$$W(A) = \{\xi^* A\xi : \xi \in \mathbb{C}^n, \xi^*\xi = 1\}.$$

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The range $W(A)$ is a convex set due to the Toeplitz–Hausdorff theorem [12]. The numerical range and its generalizations have been extensively studied for the last few decades. In particular, the numerical ranges of typical classes of matrices have attracted many attention. One of the investigated classes of matrices is the class of unitary bordering matrices. An $n \times n$ matrix A is called a unitary bordering matrix (or completely non-unitary contraction with defect index 1) if A is a contraction, that is, $\langle A^*A\xi, \xi\rangle \le \langle \xi, \xi\rangle$ for $\xi \in \mathrm{C}^n$, the modulus of any eigenvalue of A is strictly less than 1, and all but one of the singular values of A equal one. The entries of a standard form of a unitary bordering matrix $A = (a_{ij})$ in the upper triangular form, up to unitary equivalence, are determined by its eigenvalues $a_1, a_2, \dots, a_n$ in the following way

$$a_{ij} = \begin{cases} a_i & \text{if } i = j \\ \left(\prod_{k=i+1}^{j-1}(-\overline{a_k})\right)\sqrt{(1-|a_i|^2)(1-|a_j|^2)} & \text{if } i < j \\ 0 & \text{if } i > j \end{cases}. \tag{1.1}$$

The upper triangular standard form (1.1) of A can be found in [17, Theorem 4] or [11, page 180]. Mirman [17] and Gau–Wu [10] independently proved that the boundary $\partial W(A)$ of the numerical range $W(A)$ of an $n \times n$ unitary bordering matrix A and the unit circle have the Poncelet property, i.e., for any point λ on the unit circle, there is a unique $(n+1)$-gon having λ as one vertex which is circumscribed to $\partial W(A)$ and inscribed in the unit circle (cf. [1,3,11,18]).

From the Kippenhahn's viewpoint [14], the numerical range $W(A)$ of an $n \times n$ matrix A is characterized as the convex hull of the real affine part of the dual projective curve of $F_A(x, y, z) = 0$, where the determinantal ternary form associated to A is defined by

$$F_A(x, y, z) = \det(x\,\Re(A) + y\,\Im(A) + z\,I_n),$$

and $\Re(A) = (A + A^*)/2$, $\Im(A) = (A - A^*)/(2i)$. A ternary form $F(x, y, z)$ is said to be *hyperbolic* with respect to $(0, 0, 1)$ if the equation $F(x_0, y_0, z) = 0$ has only real roots in z for any $(x_0, y_0) \in \mathbb{R}^2$, and $F(0, 0, 1) = 1$. Obviously, the ternary form $F_A(x, y, z)$ is hyperbolic. In 1981, Fiedler [8] conjectured the converse is true, namely, every hyperbolic ternary form admits a determinantal ternary form. Recently, Plaumann and Vinzant [20] proved that a hyperbolic ternary form $F(x, y, z)$ of degree n satisfying $F(0, 0, 1) = 1$ admits a derminantal representation

$$F(x, y, z) = \det(xH + yK + zI_n)$$

via Hermitian matrices H, K. Their proof is rather elementary. In [16], Lentzos and Pasley solved affirmatively a related problem for a special class of hyperbolic forms raised in [5]. Historically, Lax [15] conjectured an assertion more strict than that of Fiedler in the sense that a hyperbolic ternary form $F(x, y, z)$ admits a derminantal representation