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Multiplicities of distance Laplacian eigenvalues and forbidden subgraphs

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In this work, the graphs of order n having the second distance Laplacian eigenvalue of multiplicity $n-2$ are determined. Besides that, this result also characterizes the graphs where the multiplicity of some distance Laplacian eigenvalue is equal to $n-2$. In addition, all connected graphs of order n where the largest eigenvalue of the distance Laplacian matrix has multiplicity $n-3$ are determined. Finally, we determine some graphs with a distance Laplacian eigenvalue having multiplicity $n-3$.

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Laplacian matrix
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1. Introduction

Let $G = (V, E)$ be a connected graph of order n and let $d_{i,j}$ be the distance (the length of a shortest path) between vertices v_i and v_j of G. The distance matrix of G, denoted by $\mathcal{D}(G)$, is the $n \times n$ matrix whose (i, j)-entry is equal to $d_{i,j}$, for $i, j = 1, 2, \ldots, n$. For $1 \leq i \leq n$, the sum of the distances from v_i to all other vertices in G is known as the transmission of the vertex v_i and is denoted by $Tr(v_i)$, [3]. Let $Tr(G)$ be the diagonal matrix of order n whose (i, i)-entry is equal to $Tr(v_i)$. This matrix is known as the transmission matrix of G. The Laplacian for the distance matrix of G, $\mathcal{D}^L(G)$, was introduced by M. Aouchiche and P. Hansen in [1] and it is the difference between the transmission matrix and the distance matrix, that is, $\mathcal{D}^L(G) = Tr(G) - \mathcal{D}(G)$. This matrix is known as the distance Laplacian matrix and it is a positive semidefinite matrix. Let $(\partial_1^L(G), \partial_2^L(G), \ldots, \partial_n^L(G) = 0)$ be the distance Laplacian spectrum of the connected graph G, denoted by $\mathcal{D}^L(G)$-spectrum, where $\partial_1^L(G) \geq \partial_2^L(G) \geq \ldots \geq \partial_n^L(G) = 0$. The multiplicity of the eigenvalue $\partial_i^L(G)$, $i = 1 \ldots, n$, is denoted by $m(\partial_i^L(G))$. Recall that $\partial_{n-1}^L(G) = n$ if and only if $\overline{G}$, the complement of G, is disconnected. Moreover, $\partial_{n-1}^L(G) \geq n$ and the multiplicity of n as an eigenvalue of $\mathcal{D}^L(G)$ is one less than the number of components of $\overline{G}$, [1]. More results about the distance Laplacian matrix can be found in [2,5,7,8].

In [2], M. Aouchiche and P. Hansen proposed some conjectures involving the distance Laplacian matrix. Among them, C. da Silva Jr. et al. [5] solved one in the following theorem:

Theorem 1.1 *([5]). If G is a graph on $n \geq 3$ vertices and $G \ncong K_n$, then $m(\partial_1^L(G)) \leq n-2$ with equality if and only if G is the star S_n or the complete bipartite graph $K_{p,p}$, if $n = 2p$.*

In Section 3 we characterize the graphs such that the second distance Laplacian eigenvalue has multiplicity equal to $n-2$. This result also completely characterizes the graphs where the multiplicity of some distance Laplacian eigenvalue is equal to $n-2$, extending Theorem 1.1.

In Section 4 we answer the question posed in [5] about which graphs G on n vertices have $m(\partial_1^L(G)) = n-3$. For this, we also prove that $G \cong K_{n-2} \vee \overline{K_2}$, the complete graph minus an edge, is determined by the multiplicities of its Laplacian eigenvalues. Finishing this section, we investigate the graphs for which another distance Laplacian eigenvalue has multiplicity $n - 3$, addressing the cases where n is an eigenvalue for this matrix.

2. Preliminaries

In what following, $G = (V, E)$, or just G, denotes a graph with n vertices and $\overline{G}$ denotes its complement. The diameter of a connected graph G is denoted by $diam(G)$.