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The method of Gauss–Newton to compute power series solutions of polynomial homotopies ☆

Nathan Bliss, Jan Verschelde *

University of Illinois at Chicago, Department of Mathematics, Statistics, and Computer Science, 851 S. Morgan Street (m/c 249), Chicago, IL 60607-7045, USA

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We consider the extension of the method of Gauss–Newton from complex floating-point arithmetic to the field of truncated power series with complex floating-point coefficients. With linearization we formulate a linear system where the coefficient matrix is a series with matrix coefficients, and provide a characterization for when the matrix series is regular based on the algebraic variety of an augmented system. The structure of the linear system leads to a block triangular system. In the regular case, solving the linear system is equivalent to solving a Hermite interpolation problem. We show that this solution has cost cubic in the problem size. In general, at singular points, we rely on methods of tropical algebraic geometry to compute Puiseux series. With a few illustrative examples, we demonstrate the application to polynomial homotopy continuation.

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* Corresponding author.
E-mail addresses: nbliss2@uic.edu (N. Bliss), janv@uic.edu (J. Verschelde).
URL: http://www.math.uic.edu/~jan (J. Verschelde).

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1. Introduction

1.1. Preliminaries

A polynomial homotopy is a family of polynomial systems which depend on one parameter. Numerical continuation methods to track solution paths defined by a homotopy are classical, see e.g.: [3] and [27]. Studies of deformation methods in symbolic computation appeared in [10], [11], and [17]. In particular, the application of Padé approximants in [22] stimulated our development of methods to compute power series.

Problem statement. We want to define an efficient, numerically stable, and robust algorithm to compute a power series expansion for a solution curve of a polynomial homotopy. The input is a list of polynomials in several variables, where one of the variables is a parameter denoted by t, and a value of t near which information is desired. The output of the algorithm is a tuple of series in t that vanish up to a certain degree when plugged in to either the original equations or, in special cases, a transformation of the original equations.

A power series for a solution curve forms the input to the computation of a Padé approximant for the solution curve, which will then provide a more accurate predictor in numerical path trackers. Polynomial homotopies define deformations of polynomial systems starting at generic instances and moving to specific instances. Tracking solution paths that start at singular solutions is not supported by current numerical polynomial homotopy software systems. At singular points we encounter series with fractional powers, Puiseux series.

Background and related work. As pointed out in [7], polynomials, power series, and Toeplitz matrices are closely related. A direct method to solve block banded Toeplitz systems is presented in [12]. The book [6] is a general reference for methods related to approximations and power series. We found inspiration for the relationship between higher-order Newton–Raphson iterations and Hermite interpolation in [24]. The computation of power series is a classical topic in computer algebra [16]. In [4], new algorithms are proposed to manipulate polynomials by values via Lagrange interpolation.

The Puiseux series field is one of the building blocks of tropical algebraic geometry [26]. For the leading terms of the Puiseux series, we rely on tropical methods [9], and in particular on the constructive proof of the fundamental theorem of tropical algebraic geometry [21], see also [23] and [28]. Computer algebra methods for Puiseux series in two dimensions can be found in [29].

Our contributions. Via linearization, rewriting matrices of series into series with matrix coefficients, we formulate the problem of computing the updates in Newton's method as a block structured linear algebra problem. For matrix series where the leading coefficient is regular, the solution of the block linear system satisfies the Hermite interpolation problem. For general matrix series, where several of the leading matrix coefficients may be rank deficient, Hermite–Laurent interpolation applies. We characterize when these cases occur using the algebraic variety of an augmented system. To solve the block

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