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On quaternionic numerical ranges with respect to nonstandard involutions

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ABSTRACT

Let ϕ be a nonstandard involution on the set of all quaternions, and the quaternion α be such that $\phi(\alpha) = \alpha$. The notion of numerical range of an $n \times n$ quaternion matrix A with respect to ϕ was introduced by Leiba Rodman (2014) [8] as

$$W_\phi^{(\alpha)}(A) = \{x_\phi Ax \ : x \text{ is an } n \times 1 \text{ quaternion vector and } x_\phi x = \alpha\},$$

where for $x = [x_1 \ \cdots \ x_n]^T$, $x_\phi = [\phi(x_1) \ \cdots \ \phi(x_n)]$. In this paper, some algebraic and geometrical properties of $W_\phi^{(0)}(.)$ for every arbitrary quaternion matrix are investigated. Moreover, a description of this set is given for 2×2 quaternion matrices, and $W_\phi^{(0)}(.)$ is characterized for ϕ-hermitian and ϕ-skewhermitian quaternion matrices. To illustrate the main results, some examples are also given.

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1. Introduction

Let $\mathbb{H}$ be the four-dimensional algebra of all quaternion numbers over the field of real numbers $\mathbb{R}$. Quaternions have useful applications in control systems, quantum mechanics, computer graphics, algebra, analysis and geometry; see [1,4,6,7,11,12]. An ordered triple (q_1, q_2, q_3) of quaternions is said to be a units triple if:

$$q_1^2 = q_2^2 = q_3^2 = -1,$$
$$q_1q_2 = q_3 = -q_2q_1, \quad q_2q_3 = q_1 = -q_3q_2, \quad q_3q_1 = q_2 = -q_1q_3, \quad \text{and}$$
$$1q = q1 = q \quad \text{for all } q \in \{q_1, q_2, q_3\}.$$

For example, the standard triple (i, j, k) is a units triple of quaternions. It is known, e.g., see [8, Proposition 2.4.2], that an ordered triple (q_1, q_2, q_3) of quaternions is a units triple iff there exists a 3×3 real orthogonal matrix $P = [p_{\alpha,\beta}]_{\alpha,\beta=1}^3$ such that $det(P) = 1$ and $q_\alpha = p_{1,\alpha}i + p_{2,\alpha}j + p_{3,\alpha}k$, where $\alpha = 1, 2, 3$. In particular, for every units triple (q_1, q_2, q_3) of quaternions, $\{1, q_1, q_2, q_3\}$ is a basis of $\mathbb{H}$. So, every $x \in \mathbb{H}$ can be uniquely written as $x = x_0 + x_1q_1 + x_2q_2 + x_3q_3$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$. It is easy to see that $|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$.

A map $\phi : \mathbb{H} \longrightarrow \mathbb{H}$ is called an involution if $\phi(x+y) = \phi(x)+\phi(y)$, $\phi(xy) = \phi(y)\phi(x)$ and $\phi(\phi(x)) = x$ for all $x, y \in \mathbb{H}$. It is clear that ϕ is one-to-one and onto. Moreover, the 4×4 matrix representing of ϕ, with respect to the standard basis of $\mathbb{H}$, is $diag(1, T)$, where $T = -I$ or T is a 3×3 real orthogonal symmetric matrix with eigenvalues $1, 1, -1$. If $T = -I$, then ϕ is the standard conjugation, and for the latter case, ϕ is called a nonstandard involution; see [8, Definition 2.4.5]. The set of all quaternions that are invariant by ϕ is denoted by $Inv(\phi)$; i.e.,

$$Inv(\phi) = \{x \in \mathbb{H} : \phi(x) = x\}.$$

Proposition 1.1. *([8, Theorem 2.5.1]) If ϕ is a nonstandard involution and $0 \neq \alpha \in Inv(\phi)$, then for every $\lambda \in Inv(\phi)$, there exists $\beta \in Inv(\phi)$ such that $\phi(\beta)\alpha\beta = \lambda$.*

Let $\mathbb{H}^n$ be the set of all n-column vectors with entries in $\mathbb{H}$, and $\mathbb{M}_n(\mathbb{H})$ be the algebra of all $n \times n$ quaternion matrices. Also, for an $n \times m$ quaternion matrix B, the $m \times n$ matrix B_ϕ is obtained by applying ϕ entrywise to the transposed B^T. Leiba Rodman in his book [8], for a quaternion matrix $A \in \mathbb{M}_n(\mathbb{H})$ and $\alpha \in Inv(\phi)$, introduced the notion of numerical range of A with respect to ϕ as:

$$W_\phi^{(\alpha)}(A) = \{x_\phi Ax \ : \ x \in \mathbb{H}^n, \ x_\phi x = \alpha\}. \tag{1}$$

We know that $0 \in Inv(\phi)$. In this paper, we are going to study some algebraic and geometrical properties of $W_\phi^{(0)}(A)$. For this, in Section 2, we state some preliminaries and essential properties of $W_\phi^{(0)}(A)$ which can be found in [8]. We investigate some