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Norm inequalities for a class of elementary operators generated by analytic functions with non-negative Taylor coefficients in ideals of compact operators related to p-modified unitarily invariant norms

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ABSTRACT

Let $\sum_{n=1}^{\infty}\left(\|A_nh\|^2+\|A_n^*h\|^2+\|B_nh\|^2+\|B_n^*h\|^2\right) < +\infty$ for all h in a Hilbert space $\mathcal{H}$, for some families $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ of bounded operators on $\mathcal{H}$, where at least one of them consists of mutually commuting normal operators. If $p \geqslant 2$, Φ is a symmetrically normed (s.n.) function, $\Phi^{(p)}$ is its p-modification, $\Phi^{(p)^*}$ is a s.n. function adjoint to $\Phi^{(p)}$ and $\|\cdot\|_{\Phi^{(p)^*}}$ is a norm on the ideal $\mathfrak{C}_{\Phi^{(p)^*}}(\mathcal{H})$, associated to the s.n. function $\Phi^{(p)^*}$, then for all $X \in \mathfrak{C}_{\Phi^{(p)^*}}(\mathcal{H})$

$$\begin{aligned}&\Big\|\sum_{n=1}^{\infty} A_nXB_n\Big\|_{\Phi^{(p)^*}}\\ &\leqslant \Big\|\Big(\sum_{n=1}^{\infty} A_n^*A_n\Big)^{1/2} X \Big(\sum_{n=1}^{\infty} B_nB_n^*\Big)^{1/2}\Big\|_{\Phi^{(p)^*}}.\end{aligned} \tag{1}$$

Amongst other applications, this new Cauchy–Schwarz type norm inequality was used to explore a class of elementary operators induced by an analytic functions with non-negative

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Taylor coefficients to prove that, under conditions required for (1),

$$\left\|f\Big(\sum_{n=1}^{\infty} A_n\otimes B_n\Big)X\right\|_{\Phi^{(p)*}} \leqslant \left\|\sqrt{f\Big(\sum_{n=1}^{\infty} A_n^*\otimes A_n\Big)(I)}\,X\sqrt{f\Big(\sum_{n=1}^{\infty} B_n\otimes B_n^*\Big)(I)}\right\|_{\Phi^{(p)*}},$$

whenever $\|\sum_{n=1}^{\infty} A_n^*A_n\|$, $\|\sum_{n=1}^{\infty} A_nA_n^*\|$, $\|\sum_{n=1}^{\infty} B_n^*B_n\|$ and $\|\sum_{n=1}^{\infty} B_nB_n^*\|$ are smaller then the radius of convergence of an analytic function f, where $A_n\otimes B_n$ stands for the bilateral multipliers $A_n\otimes B_n\colon \mathfrak{B}(\mathcal{H})\to\mathfrak{B}(\mathcal{H})\colon X\mapsto A_nXB_n$. Different applications and examples for the obtained norm inequalities are also provided.

1. Introduction

Let $\mathfrak{B}(\mathcal{H})$ and $\mathfrak{C}_\infty(\mathcal{H})$ denote respectively spaces of all bounded and all compact linear operators acting on a separable, complex Hilbert space $\mathcal{H}$. Each "symmetric gauge" or "symmetrically norming" (s.n.) function Φ on sequences gives rise to a symmetric or a unitary invariant (u.i.) norm on operators defined by $\|A\|_\Phi \stackrel{\text{def}}{=} \Phi(\{s_n(A)\}_{n=1}^{\infty})$, with $s_1(A)\geqslant s_2(A)\geqslant\cdots$ being the singular values of A, i.e., the eigenvalues of $|A|\stackrel{\text{def}}{=}(A^*A)^{1/2}$. Any such norm is defined on the naturally associated norm ideal $\mathfrak{C}_\Phi(\mathcal{H})$ of $\mathfrak{C}_\infty(\mathcal{H})$ and satisfies the invariance property $\|UAV\|_\Phi=\|A\|_\Phi$ for all $A\in\mathfrak{C}_\Phi(\mathcal{H})$ and for all unitary operators $U,V\in\mathfrak{B}(\mathcal{H})$. Each $\|\cdot\|_\Phi$ is lower semicontinuous, i.e., $\|w-\lim_{n\to\infty}X_n\|_\Phi\leqslant\liminf_{n\to\infty}\|X_n\|_\Phi\stackrel{\text{def}}{=}\lim_{n\to\infty}\inf_{k\geqslant n}\|X_k\|_\Phi$. This follows from the well-known formula $\|X\|_\Phi=\sup\big\{\frac{|\mathrm{tr}(XY)|}{\|Y\|_{\Phi^*}} : Y \text{ is finite rank operator}\big\}$, where Φ^* stands for the adjoint s.n. function to Φ (see [12, th. 2.7(d)] and [2, III(11.1)]).

Some well known u.i. norms are Schatten p-norms defined as $\|A\|_p\stackrel{\text{def}}{=}\sqrt[p]{\sum_{n=1}^{\infty}s_n^p(A)}$, for $1\leqslant p<+\infty$, while $\|A\|_\infty\stackrel{\text{def}}{=}s_1(A)$ coincides with the operator norm $\|A\|$. For $0<p<1$, we have quasinorms instead. Ideals of compact operators associated to those norms will be denoted by $\mathfrak{C}_p(\mathcal{H})$. Schatten p-norms are also classical examples of (degree) p-modified norms. Namely, any u.i. norm $\|\cdot\|_\Phi$ could be p-modified for any $p>0$ by setting $\|A\|_{\Phi^{(p)}}\stackrel{\text{def}}{=}\||A|^p\|_\Phi^{1/p}$ for all $A\in\mathfrak{C}_\infty(\mathcal{H})$, such that $|A|^p\in\mathfrak{C}_\Phi(\mathcal{H})$. We refer to a s.n. function $\Phi^{(p)}$ as to a p-modified s.n. function. When $p=2$, then $\|\cdot\|_{\Phi^{(2)}}$ are also known as Q-norms. For a simple proof of the triangle inequality (when $p\geqslant 1$) and other properties of these norms, including Hölder inequality, see preliminary section in [7] or [1, cor. IV.2.6] and [1, ex. IV.2.7-8]. Hence, as $\Phi^{(p)}=\big(\Phi^{(\frac{p}{2})}\big)^{(2)}$, then $\Phi^{(p)}$ are also Q-norms for all $p\geqslant 2$. Norms $\|\cdot\|_{\Phi^{(p)*}}$ represent dual norms for $\|\cdot\|_{\Phi^{(p)}}$ and some classes of these dual norms were characterized by [7, th. 2.1].