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Horizontal and vertical formulas for exponential Riordan matrices and their applications ☆

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ABSTRACT

In this paper, we show that an infinite lower triangular matrix $A = [a_{ij}]_{i,j\in\mathbb{N}_0}$ is an exponential Riordan matrix $A = \mathcal{E}(g, f)$ given by $\sum_{i\geq j} a_{ij}z^i/i! = gf^j/j!$ if and only if there exist both a horizontal pair $\{h_n; \tilde{h}_n\}_{n\geq 0}$ and a vertical pair $\{v_n; \tilde{v}_n\}_{n\geq 0}$ of sequences that represent all the elements in the matrix. As a consequence, we obtain that if the horizontal and vertical pairs of an exponential Riordan matrix are identical then the matrix is an involution. In addition, this concept can be applied to obtain the determinants of the production matrix and some conditions for the d-orthogonality of the Sheffer polynomial sequences.

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1. Introduction

Let $\mathcal{F}_n$ be the set of exponential generating functions (egf for short) of the form

$$f_n \frac{z^n}{n!} + f_{n+1} \frac{z^{n+1}}{(n+1)!} + \cdots \in \mathbb{C}[[z]], \quad f_n \neq 0.$$

An *exponential Riordan matrix* or e-Riordan matrix [1,13] $[r_{n,k}]_{n,k\geq 0}$ is defined by a pair of formal power series $g(z) \in \mathcal{F}_0$ and $f(z) \in \mathcal{F}_1$ such that its kth column egf is $\sum_{n\geq k} r_{n,k} z^n/n! = g(z)f^k(z)/k!$ or its (n,k)-entry is $r_{n,k} = n!/k![z^n]gf^k$ where $[z^n]$ is the coefficient extraction operator. As usual, the matrix is denoted by $\mathcal{E}(g(z), f(z))$ or $\mathcal{E}(g,f)$. Let $\mathcal{E}$ be the set of all e-Riordan matrices. The set $\mathcal{E}$ forms a group called the *exponential Riordan group* under the Riordan multiplication defined by

$$\mathcal{E}(g,f) * \mathcal{E}(h,\ell) = \mathcal{E}(g \cdot h(f), \ell(f)).$$

The identity is $\mathcal{E}(1,z)$, the usual identity matrix and $\mathcal{E}(g,f)^{-1} = \mathcal{E}(1/g(\overline{f}), \overline{f})$ where $\overline{f}$ is the compositional inverse of f, *i.e.*, $\overline{f}(f(z)) = f(\overline{f}(z)) = z$.

It may be also shown that if we multiply $\mathcal{E}(g,f)$ by a column vector $(\pi_0, \pi_1, \ldots)^T$ with $\phi(z) = \sum_{n\geq 0} \pi_n \frac{z^n}{n!}$ then the resulting column vector $(b_0, b_1, \ldots)^T$ has the egf $g(z)\phi(f(z)) = \sum_{n\geq 0} b_n \frac{z^n}{n!}$. We call this the *fundamental theorem* of Riordan matrices (FTRM), and we write this simply as

$$\mathcal{E}(g,f)\phi = g \cdot \phi(f). \tag{1}$$

The importance of the e-Riordan group is underlined by the fact that well-known combinatorial sequences such as the Stirling numbers of both kinds, Lah numbers, Bessel numbers etc. can be expressed as e-Riordan matrices. Moreover, e-Riordan matrix methods give simple proofs of their identities. Thus the e-Riordan group has been studied combinatorially [1,3,4,6].

A polynomial sequence $\{P_n(x)\} \subset \mathcal{P}$ is called *Sheffer* for (g,f) if there exists a pair of functions $g \in \mathcal{F}_0$ and $f \in \mathcal{F}_1$ such that

$$ge^{xf} = \sum_{n\geq 0} P_n(x) \frac{z^n}{n!}$$

where $P_0(x) = 1$. Since $(P_0(x), P_1(x), \ldots)^T = [p_{n,k}](1, x, x^2, \ldots)^T$, it follows from (1) that the sequence $\{P_n(x)\}$ is Sheffer for (g,f) if and only if the coefficient matrix $[p_{n,k}]$ is the e-Riordan matrix given by $\mathcal{E}(g,f)$. For instance, Hermite, Laguerre, Poisson–Charlier and Meixner polynomial sequences are well-known Sheffer sequences and they are orthogonal. Thus e-Riordan matrices have been studied in the context of the d-orthogonality of the Sheffer polynomial sequences [1,2,7].