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A semi-analytical approach for the positive semidefinite Procrustes problem ☆

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ABSTRACT

The positive semidefinite Procrustes (PSDP) problem is the following: given rectangular matrices X and B, find the symmetric positive semidefinite matrix A that minimizes the Frobenius norm of $AX - B$. No general procedure is known that gives an exact solution. In this paper, we present a semi-analytical approach to solve the PSDP problem. First, we characterize a family of positive semidefinite matrices that either solve the PSDP problem when the infimum is attained or give arbitrary accurate approximations to the infimum when it is not attained. This characterization requires the unique optimal solution of a smaller PSDP problem where B is square and X is diagonal with positive diagonal elements. Second, we propose a very efficient strategy to solve the PSDP problem, combining the semi-analytical approach, a new initialization strategy and the fast gradient method. We illustrate the effectiveness of the new approach, which is guaranteed to converge linearly, compared to state-of-the-art methods.

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1. Introduction

Given X, $B \in \mathbb{R}^{n,m}$, the positive semidefinite Procrustes (PSDP) problem is defined by

$$\inf_{A \in \mathcal{S}_{\succeq}^{n}} \|AX - B\|_F^2, \qquad (\mathcal{P})$$

where $\|\cdot\|_F$ is the Frobenius norm of a matrix, $\mathbb{R}^{n,r}$ the set of $n \times r$ real matrices with the special case $\mathbb{R}^n = \mathbb{R}^{n,1}$, and $\mathcal{S}_{\succeq}^{n}$ the set of symmetric positive semidefinite matrices of size n.

This problem occurs for example in the field of structure analysis [7] and in signal processing [27]. For an elastic structure, each column of X consists of generalized forces while each column of B consists of the corresponding displacements, for a set of m-measurements. From this data, it is possible to recover the so-called compliance matrix A that relates these column vectors by $AX = B$ and that must be symmetric positive definite. Such a compliance matrix may not exist for the available measurements and it is therefore desirable to find the matrix A that solves $(\mathcal{P})$ instead [32]. Solutions to $(\mathcal{P})$ can also be used when looking for the nearest stable matrix to an unstable one using a block-coordinate descent method [11,10]. This is what initially motivated us to study this problem.

In the simplest case when X is the identity matrix, the nearest positive semidefinite matrix to B in the Frobenius norm is given by $(C + H)/2$, where H is the symmetric polar factor of the matrix $C = (B + B^T)/2$ [15]. Equivalently, the projection $\mathcal{P}_{\succeq}(B)$ of B onto the cone of semidefinite matrices is given by

$$\mathcal{P}_{\succeq}(B) = U \left(\max\left(\Gamma, 0\right)\right) U^T, \qquad (1)$$

where $U\Gamma U^T$ is an eigenvalue decomposition of the symmetric matrix $\frac{B+B^T}{2}$.

The problem of finding the nearest Hermitian positive semidefinite matrix with a Toeplitz structure is studied in [27]. If the feasible set in $(\mathcal{P})$ is chosen to be the set of orthogonal matrices, then the problem is called the orthogonal Procrustes problem which arises in many applications such as computer vision, factor analysis, multidimensional scaling, and manifold optimization; see [12,14,26,1] and references therein. On the other hand, the symmetric Procrustes problem, where the feasible set in $(\mathcal{P})$ is the set of symmetric matrices arise in applications such as determination of space craft altitudes and the stiffness matrix of an elastic structure [16,7,21].

In a recent work, Alam and Adhikari [2] have characterized and determined all solutions of the structured Procrustes problem analytically, where the feasible set in $(\mathcal{P})$ is either a Jordan algebra or a Lie algebra associated with an appropriate scalar product on $\mathbb{R}^n$ or $\mathbb{C}^n$.

The theoretical and computational aspects of PSDP problem have been extensively studied in the literature. It was first introduced and studied by Allwright [4], and later in