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Maximizing the spectral radius of graphs with fixed minimum degree and edge connectivity

Wenjie Ning [a,*], Mei Lu [b], Kun Wang [c]

[a] *College of Science, China University of Petroleum (East China), Qingdao 266580, China*
[b] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*
[c] *School of Mathematical Sciences, Anhui University, Hefei 230601, China*

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A B S T R A C T

The spectral radius $\rho(G)$ of a graph G is the largest eigenvalue of the adjacency matrix $A(G)$. Suppose a graph G_0 maximizes the spectral radius over the class of graphs of order n with fixed minimum degree δ and edge connectivity $\kappa' < \delta$. In this paper, we mainly show that $G_0 \cong B^{\kappa'}_{n,\delta}$, where $B^{\kappa'}_{n,\delta}$ is obtained by adding κ' edges between $K_{\delta+1}$ and $K_{n-\delta-1}$. A property of the adjacency matrix of G_0 is also obtained. Moreover, graphs that maximize $\rho(G)$ over the class of graphs with minimum degree δ and edge-connectivity κ', for $\kappa' = 0, 1, 2, 3, \delta$, are completely determined.

* Corresponding author.
E-mail addresses: ningwenjie-0501@163.com (W. Ning), mlu@math.tsinghua.edu.cn (M. Lu), wangkun26@163.com (K. Wang).

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1. Introduction

In this paper, a graph means a simple undirected graph. Let $G = (V, E)$ be a graph of order n with vertex set $V = \{v_1, v_2, \dots, v_n\}$. Let $N_G(v)$ or for short $N(v)$ be the set of vertices adjacent to v in G and $N[v] = N(v) \cup \{v\}$. The *degree* of v in G, denoted by $d_G(v)$, is equal to $|N_G(v)|$. Denote by $\delta(G)$ or for short δ the minimum degree of vertices in G. If A and B are two disjoint subsets of V, then $[A, B] := \{uv \in E(G) | u \in A, v \in B\}$. For a subset S of V, let $N(S) = \bigcup_{v \in S} N(v)$ and denote by $G[S]$ the subgraph of G induced by S. G is *connected* if each pair of vertices is joined by a path. Suppose $U \subseteq E$. Then U is an *edge cut* of G if $G - U$ is disconnected. The *edge connectivity* of G, denoted by $\kappa'(G)$ or κ', is the minimum cardinality of an edge cut of G. Obviously, we have $\kappa' \leq \delta$. If $\kappa' = \delta$, then G is said to be *maximally edge-connected*.

The *adjacency matrix* of a graph G is the matrix $A(G) = (a_{ij})$, where a_{ij} equals 1 if v_i, v_j are adjacent and equals 0 otherwise. It is obvious that $A(G)$ is a real symmetric matrix. Thus its eigenvalues are real numbers. The largest eigenvalue of $A(G)$, denoted by $\rho(G)$, is called *the spectral radius* of G. Suppose G is connected. Then $A(G)$ is a nonnegative irreducible matrix. Thus, by Perron–Frobenius theorem, we have $\rho(G) > 0$ and there exists a unique positive unit vector x called *Perron vector* such that $A(G)x = \rho(G)x$. Moreover, we have $\delta \leq \rho(G) \leq \Delta$, and the equality holds in either of these inequalities if and only if G is regular. Given a partition $\pi = (V_1, V_2, \dots, V_r)$ of $V(G)$, the *quotient matrix* $A_\pi(G)$ of G with respect to π is a $r \times r$ matrix (b_{ij}) such that b_{ij} is the average number of neighbors in V_j of the vertices in V_i for $1 \leq i, j \leq r$. The partition π is said to be *equitable* (or *regular*) if the number of neighbors in V_j of a vertex v in V_i is a constant, independent of v.

Recently there is a lot of work on the spectral radius of a graph (see, for example [2–4,7, 8,14]), especially on maximizing the spectral radius among a given set of graphs. Berman and Zhang [1] characterized the unique graph maximizing the spectral radius among graphs with fixed number of cut vertices. Liu et al. [11] found the graph maximizing the spectral radius among graphs with fixed number of cut edges. Let $K(p, q)$ $(p \geq q \geq 0)$ be a graph obtained from K_p by adding a vertex together with edges joining this vertex to q vertices of K_p. Ye et al. [13] proved that among all graphs with fixed order n and given vertex or edge connectivity r $(1 \leq r \leq n-2)$, the graph $K(n-1, r)$ has maximum spectral radius. It is still of interest to investigate the problem of maximizing the spectral radius of graphs with fixed minimum degree and edge connectivity.

For $n \geq 2$, $\delta \geq \kappa' \geq 0$, let $\mathcal{G}_{n,\delta}^{\kappa'} = \{\, G \mid G$ is a graph of order n with minimum degree δ and edge-connectivity $\kappa' \,\}$. Note that $\mathcal{G}_{n,\delta}^{0}$ consists of disconnected graphs of order n and minimum degree δ. Ye et al. [13] proved that $K_{n-1} \cup K_1$ is the unique graph with maximum spectral radius among all graphs of order n and edge connectivity $\kappa' = 0$. Since $\delta(K_{n-1} \cup K_1) = 0$, we obtain that $K_{n-1} \cup K_1$ is also the unique one with maximum spectral radius among $\mathcal{G}_{n,0}^{0}$. So we omit this case $\delta = \kappa' = 0$ in the next sections. For $n \geq 2\delta + 2, \delta > \kappa' \geq 0$, let $B_{n,\delta}^{\kappa'}$ be obtained by adding κ' edges between $K_{\delta+1}$ and $K_{n-\delta-1}$. Note that $B_{n,\delta}^{0} = K_{\delta+1} \cup K_{n-\delta-1}$. Obviously, $B_{n,\delta}^{\kappa'} \in \mathcal{G}_{n,\delta}^{\kappa'}$.