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Localization of eigenvalues of doubly cyclic matrices

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ABSTRACT

For fixed positive α and β, and a fixed integer n, $n \geq 2$, we consider the family of matrices $\operatorname{diag}(a_1, a_2, \ldots, a_n) - \operatorname{diag}(b_1, b_2, \ldots, b_n)\Sigma_*$, where all the a_k's and b_k's are positive, the geometric mean of the a_k's and b_k's must be α and β, respectively, and Σ_* denotes the permutation matrix corresponding to the cycle $(1, 2, \ldots, n)$.
C. Johnson, Z. Price, and I. Spitkovsky conjectured that in this family, the number of eigenvalues in the left half-plane is maximized by $\alpha I - \beta\Sigma_*$; we prove this conjecture. Moreover, the complete range of possibilities for the number of eigenvalues in the left half-plane is demonstrated: if $\alpha < \beta$, then any odd number between 1 and the maximum, inclusive, is attainable.

1. Introduction

For $n \in \mathbb{N}$, $n \geq 2$, we consider matrices $X \in M_n(\mathbb{R})$ of a particular form. Defining $\mathbb{R}_{>0} = (0, \infty)$, and fixing vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in $(\mathbb{R}_{>0})^n$, we study the matrix

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$$X = \begin{pmatrix} a_1 & -b_1 & 0 & \cdots & 0 \\ 0 & a_2 & -b_2 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & a_{n-1} & -b_{n-1} \\ -b_n & 0 & \cdots & 0 & a_n \end{pmatrix}. \tag{1.1}$$

Since

$$\det X = \alpha^n - \beta^n, \quad \alpha \equiv \left(\prod_{k=1}^n a_k\right)^{1/n}, \quad \beta \equiv \left(\prod_{k=1}^n b_k\right)^{1/n} \tag{1.2}$$

the geometric means of the a_k's and b_k's play a key role. We let $DC(\alpha, \beta)$ denote the set of matrices of the form (1.1) with given geometric mean α for the a_k's and β for the b_k's.

Inspired by the occurrence of such matrices in the previous paper [1], C. Johnson, Z. Price, and I. Spitkovsky, in [2], consider the number of eigenvalues of such a matrix in the left half-plane. In particular, they note that for several cases (when $n \le 4$, or $\cos\left(\frac{2\pi}{n}\right) < \frac{\alpha}{\beta} < 1$), the number of eigenvalues in the left-half-plane is the same as that for $\alpha I - \beta\Sigma_*$. Here, I is the identity $n \times n$ matrix and

$$\Sigma_* = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{1.3}$$

is the relevant permutation $n \times n$ matrix.

Numerical evidence presented in [2] suggests that in general, the number of eigenvalues in the left half-plane for any matrix in $DC(\alpha, \beta)$ is bounded above by the corresponding value for $\alpha I - \beta\Sigma_*$. In this paper, we prove this conjecture.

Theorem 1.1. *Fix $n \in \mathbb{N}$, $n \ge 2$. Fix $\alpha, \beta \in \mathbb{R}_{>0}$. Let $X \in DC(\alpha, \beta)$. Then the number of eigenvalues of X with negative real part does not exceed the number of eigenvalues of $\alpha I - \beta\Sigma_*$ with negative real part, and setting $X = \alpha I - \beta\Sigma_* \in DC(\alpha, \beta)$ allows us to attain this upper bound as a maximum among all elements of $DC(\alpha, \beta)$.*

See also Remark 2.2 for an extension of the claim of this theorem.

Conjugating with nonsingular matrices preserves the spectrum. We conjugate X with the diagonal matrix