Linear Algebra and its Applications 540 (2018) 203–220

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

Structural properties of minimal strong digraphs versus trees ☆

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A R T I C L E   I N F O

Article history:
Received 25 November 2016
Accepted 25 November 2017
Available online 2 December 2017
Submitted by R. Brualdi

MSC:
05C20
05C05
05C75
05C40
05C70

Keywords:
Minimal strong digraphs
Trees
Characteristic polynomial
Nearly reducible matrices

A B S T R A C T

In this article, we focus on structural properties of minimal strong digraphs (MSDs). We carry out a comparative study of properties of MSDs versus (undirected) trees. For some of these properties, we give the matrix version, regarding nearly reducible matrices. We give bounds for the coefficients of the characteristic polynomial corresponding to the adjacency matrix of trees, and we conjecture bounds for MSDs. We also propose two different representations of an MSD in terms of trees (the union of a spanning tree and a directed forest; and a double directed tree whose vertices are given by the contraction of connected Hasse diagrams).

☆ Partially supported by FEDER-MINECO, MTM2015-65764-C3-1-P.

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https://doi.org/10.1016/j.laa.2017.11.027
0024-3795/

1. Introduction

A digraph is *strongly connected* or (simply) *strong* (SD) if every pair of vertices are joined by a directed path. An SD is *minimal* (MSD) if it loses the strong connection property when any of their arcs is suppressed. This class of digraphs has been considered under different points of view. See, for instance, [2,5,6,8,10–15].

It is well known that a digraph is SD if and only if its adjacency matrix is irreducible [6]. The set of SDs of order n with vertex set V can be partially ordered by the relation of inclusion among their sets of arcs. Then, the MSDs are the minimal elements of this partially ordered set. Analogously, the set of irreducible $(0, 1)$-matrices of order n with zero trace can be partially ordered by means of the coordinatewise ordering. The minimal elements of this partially ordered set are *nearly reducible matrices*. Hence, nearly reducible matrices are irreducible matrices which cease to be so if we make any of their 1-entries zero, and so a digraph is an MSD if and only if its adjacency matrix is a nearly reducible matrix [6,15]. Hartfiel [13] gives a remarkable canonical form for nearly reducible matrices.

We are also interested in the following nonnegative inverse eigenvalue problem [18]: given real numbers $k_1, k_2, \ldots, k_n$, find necessary and sufficient conditions for the existence of a nonnegative matrix A of order n with characteristic polynomial $x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_n$. The coefficients of the characteristic polynomial are closely related to the cycle structure of the weighted digraph with adjacency matrix A by means of the Theorem of the coefficients [7], and the irreducible matricial realizations of the polynomial are identified with strongly connected digraphs [6]. The class of strong digraphs can easily be reduced to the class of minimal strong digraphs, so we are interested in any theoretical or constructive characterization of these classes of digraphs.

In [10], a sequentially generative procedure for the constructive characterization of the classes of MSDs is given. In addition, algorithms to compute unlabeled MSDs and their isospectral classes are described. These algorithms have been implemented to calculate the said classes of digraphs up to order 15, classified by their order and size [20]. We are also interested in properties regarding the spectral structure of this class of digraphs, mainly about the coefficients of the characteristic polynomial.

MSDs can be seen as a generalization of trees, as we pass from simple graphs to directed graphs. Although the structure of MSDs is much richer than that of trees, many analogies remain between the properties of both families. Other properties, nevertheless, undergo radical changes when passing from trees to MSDs.

In this article, we focus on structural properties of MSDs. We carry out a comparative study of properties of MSDs versus trees. For some of the properties, we also give an interpretation in terms of nearly reducible matrices, via the adjacency matrices. We also conjecture a generalization of the bounds on the coefficients of characteristic polynomials of trees to MSDs. As a particular case, the independent coefficient of the characteristic polynomial of a tree or an MSD must be -1, 0 or 1. For trees, this means that a tree has at most one perfect matching; for MSDs, it means that an MSD has at most one