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Multiplicity lists for symmetric matrices whose graphs have few missing edges

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ABSTRACT

We characterize the possible lists of multiplicities occurring among the eigenvalues of real symmetric (or Hermitian) matrices whose graph is one of K_n, K_n less an edge, or both possibilities for K_n less two edges. The lists are quite different from those for trees. Some construction techniques are developed here and additional results with more missing edges are given, including the case of several independent edges.

1. Introduction

Let G be an (undirected) simple graph on n vertices and $\mathcal{S}(G)$ be the collection of all n-by-n real symmetric matrices, the graph of whose (nonzero) off-diagonal entries is G.

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No restriction is placed by G upon the diagonal entries of $A \in \mathcal{S}(G)$. We are interested in all possible lists of multiplicities for the eigenvalues of matrices in $\mathcal{S}(G)$. Let $\mathcal{L}(G)$ be the set of all such lists. Since the total of the multiplicities is n, view these as partitions of n.

It is natural to consider connected graphs G, and in the minimally connected case of trees, the possible lists $\mathcal{L}(G)$ have been heavily studied [6–11] and have much special structure. However, a complete characterization is known only for some classes of trees.

We are interested here in the case in which G has few missing edges (the other extreme from trees), i.e., G is the complete graph K_n, or K_n with a few edges deleted. Of course, the maximum possible multiplicity, $\mathrm{M}(G)$, occurring in $\mathcal{L}(G)$ is an important constraint on these lists. Since, for symmetric matrices, algebraic multiplicity equals geometric multiplicity, $M(G) = n - mr(G)$, in which $mr(G)$ is the smallest rank occurring among matrices in $\mathcal{S}(G)$.

In general, $mr(G)$ is difficult to know, but, fortunately, when there are just a few edges missing from K_n, it is not hard to determine.

In the case of trees, there is a nice characterization of $mr(G)$ [6], but there are many additional constraints on $\mathcal{L}(G)$, such as at least two eigenvalues of multiplicity 1.

The case of high edge-density graphs seems to be in strong contrast to trees in several ways. Besides the $mr(G)$ constraint, there are often, but not always, no other constraints, and subject to the possible multiplicities, any eigenvalues are often possible. i.e., the inverse eigenvalue problem (IEP) is equivalent to the multiplicity list problem. It is an interesting question for which graphs 1) $\mathcal{L}(G)$ is all lists allowed by $mr(G)$ and 2) the IEP for G is equivalent to the $\mathcal{L}(G)$ problem for G. When this occurs for the graphs we study, we make note of it.

2. Useful tools

We identify the edges missing from K_n by the graph that they, together with their vertices, form. So, for a graph H on no more than n vertices, by $K_n - H$ we mean that the edges (only) of H are deleted from K_n. Let S_k denote the star on k vertices, and P_k the path on k vertices. We give certain graphs special names based on what is missing: $G_0 = K_n$, $G_1 = K_n$ missing one edge, $G_2 = K_n - S_3$, $G_{1,1} = K_n$ less two independent edges. More generally, $G_k = K_n - S_{k+1}$ and $G_{1,1,\ldots,1} = K_n$ less k independent edges if there are k 1's. The matrices in $\mathcal{S}(G_k)$ have all their 0 entries in one row and column, and the matrices in $\mathcal{S}(G_{1,1,\ldots,1})$ look like

$$\begin{bmatrix} * & 0 & & & & & & & & \\ 0 & * & & & & & \Huge{*} & & & \\ & & * & 0 & & & & & & \\ & & 0 & * & & & & & & \\ & & & & * & 0 & & & & \\ & & & & 0 & * & & & & \\ & & & & & & \ddots & & & \\ & & \Huge{*} & & & & & & * & \\ & & & & & & & & & * \end{bmatrix}.$$