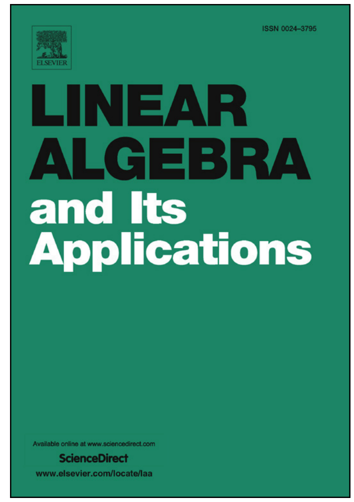
Accepted Manuscript

When does Ando–Hiai inequality hold?

Shuhei Wada



PII: S0024-3795(17)30658-4
DOI: https://doi.org/10.1016/j.laa.2017.11.030
Reference: LAA 14397

To appear in: *Linear Algebra and its Applications*

Received date: 15 August 2017
Accepted date: 28 November 2017

Please cite this article in press as: S. Wada, When does Ando–Hiai inequality hold?, *Linear Algebra Appl.* (2018), https://doi.org/10.1016/j.laa.2017.11.030

WHEN DOES ANDO–HIAI INEQUALITY HOLD?

SHUHEI WADA

Abstract. Let α be in $(0,1)$ and $r > 0$ and $\#_\alpha$ stand for the weighted operator geometric mean. We consider the following statement:

$$A, B > 0, A\#_\alpha B \geq I \Rightarrow A^r \#_\alpha B^r \geq I.$$

Ando and Hiai show that if $r \geq 1$, then this holds. In the present paper, we first prove the converse of this result, namely, the above statement holds only if $r \geq 1$. We next show that for each non-negative continuous function f on $[0, \infty)$ with $f \leq t^r$ and $f \neq t^r$, there exist $A, B > 0$ such that $A\#_\alpha B \geq I$ and $f(A)\#_\alpha f(B) \ngeq I$. We also try to find a characterization of a continuous function f satisfying

$$A, B > 0, A\#_\alpha B \geq I \Rightarrow f(A)\#_\alpha f(B) \geq I.$$

1. Introduction

Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle \cdot \mid \cdot \rangle$ and let $B(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. A bounded linear operator A is said to be positive (denoted by $A \geq 0$) if $\langle Ax \mid x \rangle \geq 0$ for all $x \in \mathcal{H}$. If a positive operator A is invertible, we denote by $A > 0$. The set of positive operators on $\mathcal{H}$ is denoted by $B(\mathcal{H})_+$.

A continuous real function f from $[0, \infty)$ is said to be operator monotone on $[0, \infty)$ if for two positive operators A and B, the inequality $A \geq B$ implies $f(A) \geq f(B)$. It is known that a non-negative operator monotone function f has the following property: for every $A, B \geq 0$, $f(\frac{A+B}{2}) \geq \frac{1}{2}f(A) + \frac{1}{2}f(B)$. Such a function is referred to as an operator concave function. A real function f, such that $-f$ is operator concave, is called operator convex. It is known that every non-constant non-negative operator convex function f on $[0, \infty)$ with $f(0) = 0$ can be written as $f(t) = th(t)$ for some operator monotone function h, so the adjoint of f, denoted by $f^*(t) \left(:= \frac{1}{f(\frac{1}{t})}\right)$ is also operator convex [2].

2010 *Mathematics Subject Classification.* Primary 47A63; Secondary 47A64.
Key words and phrases. operator mean, Ando–Hiai inequality, operator monotone.