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Bounds on the independence number and signless Laplacian index of graphs

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ABSTRACT

In this paper, we give some bounds on the signless Laplacian index of graphs in terms of independence number. In addition, these results disprove a conjecture in [3] involving the signless Laplacian index and independence number of graphs.

1. Introduction

Through this paper, we consider simple, undirected and connected graphs. Let $G = (V(G), E(G))$ be a graph. For $v \in V(G)$, the neighborhood of v, denoted by $N(v)$, is the set of all vertices in G adjacent to v. Set $d(v) = |N(v)|$ and $m(v) = \sum_{u\in N(v)} d(u)/d(v)$.

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For $X, Y \subseteq V(G)$, we let $\partial(X, Y)$ denote the set of edges of G with one end in X and the other end in Y, and let $G[X]$ be the subgraph induced by X. If $Y = V(G) \backslash X$, then we set $\partial(X) := \partial(X, Y)$ and $d(X) = |\partial(X)|$. A *clique* in a graph G is a subset C of $V(G)$ such that every two vertices in C are connected by an edge. The *clique number* of a graph G, denoted $\omega(G)$, is the number of vertices in a maximum clique of G. *An independent set* is a set of vertices in a graph, no two of which are adjacent. The *independence number* of a graph G, denoted by $\alpha(G)$, is the numbers of vertices of the largest independent set in G. Let P_n, C_n, K_n and $K_{p,n-p}$ denote the path, cycle, complete graph and complete bipartite graph of order n, respectively.

Let G and H be two disjoint graphs. The disjoint union of G and H, denoted $G + H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. If $G_1 \cong \cdots \cong G_k$, we write kG_1 for $G_1 + \cdots + G_k$.

For a graph G, let $D(G) = \mathrm{diag}(d(v_1), d(v_2), \cdots, d(v_n))$ be the diagonal matrix of vertex degrees and $A(G)$ be the adjacency matrix of G. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of the graph G. The matrix $Q(G)$ is symmetric and nonnegative, and, when G is connected, it is irreducible. Thus $Q(G)$ is positive semidefinite and its eigenvalues can be arranged as:

$$q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0.$$

The largest eigenvalue of $Q(G)$ is called the *signless Laplacian spectral radius* or the *signless Laplacian index* of $Q(G)$.

In recent years, the signless Laplacian index has received increasing attention (see [1–5], [7–11]). Recently, Hansen and Lucas [3] proposed some conjectures of the form

$$l(n) \leq q_1(G) \oplus i(G) \leq u(n),$$

where $\oplus$ is one the four operations $+, -, \times, /$ and $i(G)$ is another invariant chosen among diameter, radius, girth, independence number, clique number, chromatic number and so on. Liu and Lu [7] [8] solved two conjectures (see [3], Conjectures 17–18) on the signless Laplacian index involving the diameter and radius, respectively. He, Jin and Zhang [5] disproved two conjectures (see [3], Conjectures 29 and 31) on the signless Laplacian index. The purpose of this paper is to study the following conjecture involved the independence number of graphs.

Conjecture 1.1 *([3, Conjecture 27]). Let G be a connected graph of order $n \geq 4$. Then*

$$4 + \left\lfloor \frac{n}{2} \right\rfloor \leq q_1(G) + \alpha(G), \text{ if } n \text{ is odd,} \tag{1}$$

$$2(n-1) \leq q_1(G) \cdot \alpha(G). \tag{2}$$

The bound for (1) is attained by and only by the cycle C_n when n is odd. Moreover, if n is even, then $q_1(G) + \alpha(G)$ is minimal for the graph C_n^* of order $n \geq 8$, where C_n^* is a graph obtained from two cycles of order $2\lfloor n/6 \rfloor + 1$ by linking them by a path.