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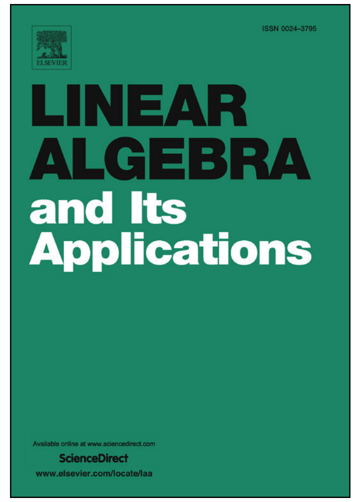

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Nodal Decompositions of Graphs

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Abstract

A nodal domain of a function is a maximally connected subset of the domain for which the function does not change sign. Courant's nodal domain theorem gives a bound on the number of nodal domains of eigenfunctions of elliptic operators. In particular, the k^{th} eigenfunction contains no more than k nodal domains. We prove a generalization of Courant's theorem to discrete graphs. Namely, we show that for the k^{th} eigenvalue of a generalized Laplacian of a discrete graph, there exists a set of corresponding eigenvectors such that each eigenvector can be decomposed into at most k nodal domains. In addition, we show this set to be of co-dimension zero with respect to the entire eigenspace.

AMS classification: 05C50

Keywords: Generalized Laplacian; Nodal Domains.

1. Introduction

In the 1923 paper titled "Ein allgemeiner Satzt zur Theorie der Eigenfunktionen selbsadjungierter Differentialausdrücke" [10], as well as in the 1924 text co-authered with David Hilbert "Methoden der mathematischen Physik I" [9], Richard Courant proved a result regarding the zeros of elliptic eigenfunctions, the so-called Courant nodal domain theorem.

Theorem 1.1 (Courant's nodal domain theorem, [10, 9])**.** *Given the self-adjoint second order differential equation $L[u] + \lambda\rho u = 0$, ($\rho \neq 0$), for a domain G with arbitrary homogeneous boundary conditions; if its eigenfunctions are ordered according to increasing eigenvalues, then the nodes of the n^{th} eigenfunction u_n divide the domain into no more than n subdomains. No assumptions are made about the number of independent variables.*

The "nodes" are the nodal set $\{x|u_n(x) = 0\}$ and the "sub-domains" are now referred to as nodal domains. Extensions of Courant's nodal domain theorem are abundant, including to p-Laplacians, Riemannian manifolds, and domains with low regularity assumptions [13, 23, 7, 11, 1]. Most notably, Pleijel's nodal domain theorem is an extension of Theorem 1.1 to vibrating membranes using Faber-Krahn results [24]. Theorem 1.1 is also closely related to the work of Chladni involving the modes of vibration of a rigid surface; the patterns of nodal lines on the surface are referred to as Chladni figures [28, 8]. For further information regarding the importance of Theorem 1.1, the author refers the reader to [2].

Courant's theorem has extensions not only in differential equations, but in graph theory as well. To see the natural extension, we note that many of the matrix representations of graphs, such as the graph Laplacian, have properties that are analogous to continuous elliptic operators.