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Ehrhart tensor polynomials

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ABSTRACT

The notion of Ehrhart tensor polynomials, a natural generalization of the Ehrhart polynomial of a lattice polytope, was recently introduced by Ludwig and Silverstein. We initiate a study of their coefficients. In the vector and matrix cases, we give Pick-type formulas in terms of triangulations of a lattice polygon. As our main tool, we introduce h^r-tensor polynomials, extending the notion of the Ehrhart h^*-polynomial, and, for matrices, investigate their coefficients for positive semidefiniteness. In contrast to the usual h^*-polynomial, the coefficients are in general not monotone with respect to inclusion. Nevertheless, we are able to prove positive semidefiniteness in dimension two. Based on computational results, we conjecture positive semidefiniteness of the coefficients in higher dimensions. Furthermore, we generalize Hibi's palindromic theorem for reflexive polytopes to h^r-tensor polynomials and discuss possible future research directions.

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1. Introduction

The Ehrhart polynomial of a lattice polytope counts the number of lattice points in its integer dilates and is arguably the most fundamental arithmetic invariant of a lattice polytope. It is a cornerstone of geometric combinatorics and appears in various guises in other areas of mathematics such as commutative algebra, optimization, representation theory, or voting theory (see, e.g., [4,7,12,24,29]). Concepts from Ehrhart theory have been generalized in various directions, for example, q-analogs of Ehrhart polynomials [11], equivariant versions [39], multivariate extensions [5,8,17], and generalizations to valuations [21,22,28].

Recently, Ludwig and Silverstein [26,34] introduced Ehrhart tensor polynomials based on discrete moment tensors that were defined by Böröczky and Ludwig [10]. Let $\mathcal{P}(\mathbb{Z}^d)$ denote the family of convex polytopes with vertices in $\mathbb{Z}^d$, called **lattice polytopes**, and let $\mathbb{T}^r$ be the vector space of symmetric tensors of rank r on $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, we write xy for $x \otimes y$. In particular, $x^r = x \otimes \cdots \otimes x$ and we set $x^0 := 1$.

The **discrete moment tensor of rank** r of a polytope $P \in \mathcal{P}(\mathbb{Z}^d)$ is

$$\mathrm{L}^r(P) = \sum_{x \in P \cap \mathbb{Z}^d} x^r \tag{1}$$

where $r \in \mathbb{N}$ and $\mathbb{N}$ denotes the set of nonnegative integers. Note that, for our convenience, this definition differs by a scalar from the original definition given in [10]. A specialization of $\mathrm{L}^r(P)$, the discrete directional moment, was studied in [33]. For $r = 0$, the usual **discrete volume** or **lattice point enumerator** $\mathrm{L}(P) := \mathrm{L}^0(P) = |P \cap \mathbb{Z}^d|$ is recovered. For $r = 1$, $\mathrm{L}^1(P)$ equals the discrete moment vector defined in [9]. Based on results by Khovanskiĭ and Pukhlikov [31] and Alesker[1], it was identified in [26] that $\mathrm{L}^r(nP)$ is given by a polynomial, for any $n \in \mathbb{N}$, extending Ehrhart's celebrated result for the lattice point enumerator [13].

Theorem *([26, Theorem 1]). There exist $\mathrm{L}_i^r : \mathcal{P}(\mathbb{Z}^d) \to \mathbb{T}^r$ for all $1 \leq i \leq d+r$ such that*

$$\mathrm{L}^r(nP) = \sum_{i=0}^{d+r} \mathrm{L}_i^r(P) n^i$$

for any $n \in \mathbb{N}$ and $P \in \mathcal{P}(\mathbb{Z}^d)$.

The expansion of $\mathrm{L}^r(nP)$ will be denoted as $\mathrm{L}_P^r(n)$ and is called the **Ehrhart tensor polynomial** of P in commemoration of this result. Furthermore, the coefficients $\mathrm{L}_1^r, \ldots, \mathrm{L}_{d+r}^r$ are the **Ehrhart tensor coefficients** or **Ehrhart tensors**.

A fundamental and intensively studied question in Ehrhart theory is the characterization of Ehrhart polynomials and their coefficients. The only coefficients that are known to have explicit intrinsic geometric descriptions are the leading, second-highest, and