Linear Algebra and its Applications 538 (2018) 22–42

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

A new characterization of subnormality for a class of 2-variable weighted shifts with 1-atomic core [☆]

Jaewoong Kim [a], Jasang Yoon [b,*]

[a] *Department of Mathematics, Korea Military Academy, Seoul, 01805, Republic of Korea*
[b] *School of Mathematical and Statistical Sciences, The University of Texas Rio Grande Valley, Edinburg, TX 78539, United States*

ARTICLE INFO

Article history:
Received 25 June 2017
Accepted 10 October 2017
Available online xxxx
Submitted by P. Semrl

MSC:
primary 47B20, 47B37, 47A13, 15B05, 15B48

Keywords:
The lifting problem for commuting subnormal operators
Subnormal pairs
Moore–Penrose inverse
2-variable weighted shifts
The Berger Theorem
The Bram–Halmos characterization
Franks' result

ABSTRACT

Given a pair $\mathbf{T} \equiv (T_1, T_2)$ of commuting subnormal Hilbert space operators, the Lifting Problem for Commuting Subnormals (LPCS) calls for necessary and sufficient conditions for the existence of a commuting pair $\mathbf{N} \equiv (N_1, N_2)$ of normal extensions of T_1 and T_2. This is an old problem in operator theory. The aim of this paper is to study LPCS. There are three well-known subnormal characterizations for operators: the Berger Theorem, the Bram–Halmos characterization, and Franks' result. In our paper, we study a new subnormal characterization which is related to these three well-known ones for a class of 2-variable weighted shifts. Thus, we can provide a large nontrivial class of 2-variable weighted shifts in which k-hyponormal (some $k \geq 1$) and subnormal are equal and the class is invariant under the action $(h, \ell) \mapsto \mathbf{T}^{(h,\ell)} := (T_1^h, T_2^\ell)$ $(h, \ell \geq 1)$.

☆ The first named author was partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2016R1D1A1A09918018). This material is based upon work supported by a grant from the University of Texas System and the Consejo Nacional de Ciencia y Tecnolog Ã-a de México (CONACYT).

* Corresponding author.
E-mail addresses: jwkim@kma.ac.kr (J. Kim), jasang.yoon@utrgv.edu (J. Yoon).

https://doi.org/10.1016/j.laa.2017.10.010
0024-3795/

1. Introduction

The *Lifting Problem for Commuting Subnormals* (LPCS) is an old problem in operator theory. The aim of this paper is to study the long-standing open problem. It is known that the subnormality of any operator can be ascertained by examining the subnormality of a specialized class of weighted shifts. Thus, our paper studies the subnormality of 2-variable weighted shifts. There are several well-known characterizations of the subnormality for weighted shifts, such as Berger Theorem, Bram–Halmos characterization, and Franks' result. To provide a concrete answer for LPCS regarding 2-variable weighted shifts, it is natural to ask if a new subnormal characterization exists which is related to these three well-known ones. This paper provides a new characterization for the subnormality of a class of 2-variable weighted shifts.

In [11, Conjecture 1], Curto, Muhly, and Xia conjectured that if $\mathbf{T} \equiv (T_1, T_2)$ is a pair of commuting subnormal operators on $\mathcal{H}$, then $\mathbf{T}$ hyponormal is enough for the lifting of $\mathbf{T}$. In [13, Theorems 2.12, 3.3, 4.3], this conjecture was answered negatively using 2-variable weighted shifts. It is known that weighted shifts have played a significant role in the study of cyclicity (or reflexivity) and C^*-algebras generated by multiplication operators on Bergman spaces (cf. [16], [22], [25], [26], [27]). Weighted shifts have also played an important role in the study of LPCS (cf. [6], [7], [8], [9], [13], [14], [15], [29]). In 1978, A.R. Lubin [24] addressed concrete problems about LPCS: if (T_1, T_2) is a commuting pair of subnormal operators, do they admit commuting normal extensions **(i)** when $p(T_1, T_2)$ is subnormal for every 2-variable polynomial p, **(ii)** when $T_1 + sT_2$ (all $s \in \mathbb{C}$) is subnormal, or more weakly, **(iii)** when $T_1 + T_2$ is subnormal? In 1994, E. Franks [17] showed that the first condition gives an affirmative answer: indeed, a commuting pair $\mathbf{T} \equiv (T_1, T_2)$, of subnormal operators T_1 and T_2, admits commuting normal extensions if $p(T_1, T_2)$ is subnormal for each 2-variable polynomial p of degree at most 5. In 2016, W.Y. Lee, S.H. Lee, and J. Yoon showed that the third condition was answered negatively using 2-variable weighted shifts in the class $\mathcal{TC}$ (see the definition given below) [23]. However, the second condition still remains a long-standing open problem. Motivated by the result of E. Franks [17], it is meaningful to study LPCS from a different approach: for $k \geq 1$ we ask to what extent k-hyponormality for the powers (T_1^h, T_2^ℓ) $(h, \ell \geq 1)$ can guarantee a lifting for (T_1, T_2). For the class of 2-variable weighted shifts, it is often the case that the powers are less complex than the initial pair; thus it becomes especially significant to unravel LPCS under the action $(h, \ell) \mapsto \mathbf{T}^{(h,\ell)} = (T_1^h, T_2^\ell)$ $(h, \ell \geq 1)$.

To describe our main results we need some notation. We use $\mathfrak{H}_0$ (resp. $\mathfrak{H}_\infty$) to denote the set of commuting pairs of subnormal operators (resp. the set of commuting subnormal pairs) on Hilbert space. For $k \geq 1$, we let $\mathfrak{H}_k$ denote the class of k-hyponormal pairs in $\mathfrak{H}_0$. Clearly, we have that

$$\mathfrak{H}_\infty \subseteq \cdots \subseteq \mathfrak{H}_k \subseteq \cdots \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_0 \text{ [6]}.$$