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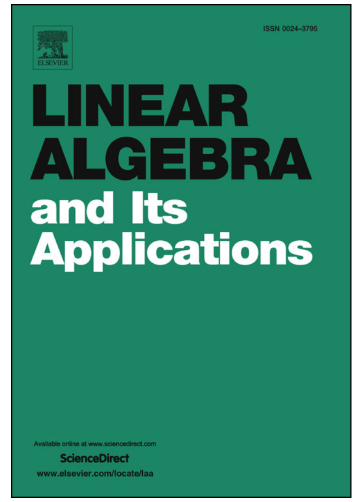

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BOUNDS ON POLYNOMIAL ROOTS USING INTERCYCLIC COMPANION MATRICES

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ABSTRACT. The Frobenius companion matrix, and more recently the Fiedler companion matrices, have been used to provide lower and upper bounds on the modulus of any root of a polynomial $p(x)$. In this paper we explore new bounds obtained from taking the 1-norm and ∞-norm of a matrix in the wider class of intercyclic companion matrices. As is the case with Fiedler matrices, we observe that the new bounds from intercyclic companion matrices can improve those from the Frobenius matrix by at most a factor of two. By using the Hessenberg form of an intercyclic companion matrix, we describe how to determine the best upper bound when restricted to Fiedler companion matrices using the ∞-norm. We also obtain a new general bound by considering the polynomial $x^q p(x)$ for $q > 0$. We end by considering upper bounds obtained from inverses of monic reversal polynomials of intercyclic companion matrices, noting that these can make more significant improvements on the bounds from a Frobenius companion matrix for certain polynomials.

1. Introduction

There are various techniques for approximating the roots of a polynomial (see for example [7, 8]). Some algorithms for determining the roots of a polynomial rely on a good first approximation (see e.g. [1]) based on the coefficients of the polynomial. One method [9] for finding the roots of a polynomial $p(x)$ is to find the eigenvalues of a companion matrix, since a companion matrix has characteristic polynomial $p(x)$. To approximate the roots of $p(x)$, one can apply Gershgorin's Theorem [9] or use matrix norms [3] on a companion matrix to find regions in the complex plane to locate the eigenvalues. For example, using these methods, one can obtain Cauchy's bound: if λ is a root of

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \tag{1}$$

then

$$|\lambda| \leq \max\{|a_0|, 1 + |a_1|, 1 + |a_2|, \ldots, 1 + |a_{n-1}|\}. \tag{2}$$

Typically one uses the classical *Frobenius companion matrix*,

$$\mathcal{F} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \tag{3}$$

for these purposes but recently other companion matrices have been discovered. In particular, the Fiedler companion matrices, introduced in [5], were explored in [3] to provide upper and lower bounds on the modulus of a root of $p(x)$. The Frobenius matrix is itself a Fiedler matrix. More recently, the sparse companion matrices (also known as intercyclic companion matrices) were characterized in [2]. This class of matrices includes the Fiedler matrices as a special case. In this paper we develop new bounds on the modulus of a root of $p(x)$ using the larger class of sparse companion matrices, comparing them with other known

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