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Biderivations of the block Lie algebras

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In this paper, we determine the biderivations of the block Lie algebras $\mathcal{B}(q)$ for all $q \in \mathbb{C}$. More precisely, we prove that the space of biderivations of $\mathcal{B}(q)$ is spanned by inner biderivations and one outer biderivation. Applying this result we also find all commuting maps on $\mathcal{B}(q)$.

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1. Introduction

Derivations and generalized derivations (including biderivations) are important concepts in algebra and have attracted a lot of attention in the related studies. See for example [1,3–9,11–15]. In [3], Brešar introduced the concept of biderivation of rings, and determined the biderivations of semiprime rings (particularly, he showed that all biderivations on noncommutative prime rings are inner). Biderivations are also very helpful tools in the study of commuting maps. In several papers of determining commuting maps, some authors first obtained all biderivations of algebras they studied. See for example [6,9,11,15].

The notion of biderivation of Lie algebras was introduced in [14] where the authors showed that skewsymmetric biderivations of finite dimensional complex simple Lie algebra are inner. Furthermore, in [11], the authors proved that biderivations (without assuming skew-symmetry) of finite dimensional complex simple Lie algebras are inner. Biderivations on the Schrödinger–Virasoro algebra and skewsymmetric biderivations of simple generalized Witt algebras over a field of characteristic 0, were shown to be inner in [13] and [6]. In [9], the authors determined all skewsymmetric biderivations of $W(a,b)$ and found some biderivations which were not inner. In [12], the biderivations without skewsymmetric condition are studied for the W-algebra $W(2,2)$ and the Virasoro algebra. Recently super-biderivations of superalgebras were studied by several authors. See for example [15–17]. More precisely, all skew-supersymmetric super-biderivations of the super-Virasoro algebra are inner in [15,16] and all super-biderivations of the centerless super-Virasoro algebras were proved to be inner in [17].

In the present paper, we will establish a new method to determine all biderivations of the block Lie algebras $\mathcal{B}(q)$ for $q \in \mathbb{C}$. Our method is conceptual that avoids a lot of computations and distincts from other papers. We believe our method will be applicable to many other Lie algebras with polynomial multiplication. As a byproduct we obtain that the biderivations on the derivation algebra $\mathcal{B}'(q)$ are inner.

Denote by $\mathbb{Z}, \mathbb{C}, \mathbb{N}$ the sets of integers, complex numbers, positive integers respectively. Now we introduce our Block Lie algebras $\mathcal{B}(q)$ for $q \in \mathbb{C}$. We always fix this q in this paper. Throughout this paper, we assume that all algebras and vector spaces are over the complex numbers $\mathbb{C}$, our results work over any algebraically closed field of characteristic zero.

Let us first recall the definition of the Block Lie algebras.

Definition 1.1. Let $q \in \mathbb{C}$. The Block Lie algebra $\mathcal{B}(q)$ is the Lie algebra with a basis $\{L_{m,i} | m, i \in \mathbb{Z}\}$ subject to the following Lie brackets

$$[L_{m,i}, L_{n,j}] = (n(i+q) - m(j+q))L_{m+n,i+j}, \ \forall\, i, j, m, n \in \mathbb{Z}.$$

This paper is organized as follows. In section 2, using results from [1] we determine all the derivations of $\mathcal{B}'(q)$ and $\mathcal{B}(q)$. In section 3, we first prove that any biderivation f on $\mathcal{B}(q)$ whose restriction on $\mathcal{B}'(q)$ is an inner biderivation must be of the form