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Analyticity and spectral properties of noncommutative Ricci flow in a matrix geometry

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ABSTRACT

We study a first variation formula for the eigenvalues of the Laplacian evolving under the Ricci flow in a simple example of a noncommutative matrix geometry, namely a finite dimensional representation of a noncommutative torus. In order to do so, we first show that the Ricci flow in this matrix geometry is analytic.

1. Introduction

In [8], Ricci flow was defined and studied in a simple example of a matrix geometry, namely in a finite dimensional representation of a noncommutative (or quantum) 2-torus. This was motivated by [2], which attempts to define Ricci flow in the usual infinite dimensional representation of the noncommutative 2-torus by using a first variation

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formula for the eigenvalues of the Laplace–Beltrami operator obtained in the classical case in [7].

In [8], however, the Ricci flow was defined more directly by a noncommutative version of the Ricci flow equation, with no reference to the spectrum of the Laplace–Beltrami operator or a first variation formula. In this paper the aim is to show that a first variation formula can in fact also be obtained for the Ricci flow as defined in [8].

The formula is obtained in Section 5. This is actually the second of the two main results of the paper. In order to prove it, we first need to show that the Ricci flow in [8] is analytic, which is the first of our main results, and is obtained in Section 3.

Section 2 briefly reviews the noncommutative Ricci flow from [8] in preparation for Section 3. The central object in Section 5 is the noncommutative Laplace–Beltrami operator. Section 4 presents this operator in the finite dimensional representation in analogy to the known Laplace–Beltrami operator in the infinite dimensional representation of the noncommutative torus.

The results of this paper contribute to showing that various properties of Ricci flow in classical (i.e. commutative) differential geometry can be systematically extended to a noncommutative example, indicating that Ricci flow can be sensibly studied in the noncommutative case. Secondly, the paper to some extent clarifies the similarities and differences between the approaches taken in [8] and [2] respectively.

As Ricci flow is of importance in differential geometry and related areas, it seems plausible that extensions of results on Ricci flow to the noncommutative case can ultimately be of value in noncommutative geometry and its applications. Keep in mind that Ricci flow originated as part of Hamilton's programme to prove the Poincaré conjecture [12], and that this programme was indeed later completed by Perelman in [24–26]. In Friedan's work at about the same time as [12], Ricci flow essentially also appeared as part of a low order approximation to the renormalization group equation of nonlinear sigma models in physics; see [10] for the initial paper, but for a clearer formulation see in particular [11, Section II.1]. These remarks clearly illustrate the power and range of applications of classical Ricci flow. Refer to [2, Section 6] for a brief discussion of the possibility of corresponding applications in the noncommutative case.

Since in the formulation used in [8] the usual partial differential equation in fact becomes a system of ordinary differential equations, one can use tools from linear algebra, systems of ordinary differential equations (including the case on a complex domain, rather than just on a real interval), complex analysis and perturbation theory of linear operators, to obtain results that in the infinite dimensional representation are far more difficult to prove or are, as yet, not accessible. An example of this is the convergence of the Ricci flow to the flat metric, shown in [8] using techniques from systems of ordinary differential equations, but which had not yet been obtained in [2]. In this paper we can use these techniques to derive the analyticity of the Ricci flow, and consequently of the eigenvalues and eigenvectors. Even in the classical case in [7], on the other hand, the existence of sufficiently smoothly parametrized eigenvalues and eigenvectors had to be