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On the trace of the Coxeter polynomial of an algebra

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ABSTRACT

Let $A = KQ/I$ be a finite dimensional triangular K-algebra. Let ϕ_A be the Coxeter matrix of A. We relate homological conditions for A with properties of the traces of the Coxeter transformation ϕ_A. For instance, a finite dimensional accessible algebra A is strongly accessible if and only if $\mathrm{Tr}\,(\phi_A) = -1$. We say A is of *cyclotomic type* if the eigenvalues of ϕ_A lie on the unit circle. Clearly, if A is of cyclotomic type then $|\mathrm{Tr}\,(\phi_A)^k| \leq n$, for $k \geq 0$. We prove that A is of cyclotomic type if $|\mathrm{Tr}\,(\phi_A)^k| \leq n$ holds for $0 \leq k \leq n$. We illustrate the results with examples of Nakayama algebras.

1. Introduction

Assume throughout the paper that K is an algebraically closed field. We assume that A is a triangular finite dimensional K-algebra, that is, $A = KQ/I$ is a quotient of the path algebra KQ associated to a quiver Q without oriented cycles. In particular, A has finite global dimension. In this situation, the bounded derived category $\mathrm{D^b}\,\mathrm{mod}_A$ has Serre duality of the form

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$$\mathrm{Hom}(X, Y[1]) = D\mathrm{Hom}(Y, \tau X)$$

where τ is a self-equivalence which serves as the Auslander–Reiten translation of $\mathrm{D^b}\,\mathrm{mod}_A$ and $D = \mathrm{Hom}_K(-, K)$ is the usual duality. The automorphism τ induces the *Coxeter transformation* ϕ_A at the Grothendieck group $K_0(A)$. This invertible matrix plays a major role in Representation Theory.

Following [7], we say that A is of *cyclotomic type* if the eigenvalues of ϕ_A lie on the unit circle. Many important finite dimensional algebras are known to be of cyclotomic type: hereditary algebras of finite or tame representation type, canonical algebras, some extended canonical algebras and many others, see [5,7]. On the other hand, there are well-known classes of algebras with a mixed behavior with respect to cyclotomicity. For instance, in Section 5 below we consider the class of *Nakayama algebras*. Let $N(n, r)$ be the quotient obtained from the linear quiver with n vertices

$$\bullet \xrightarrow{x} \bullet \xrightarrow{x} \cdots \bullet \xrightarrow{x} \bullet$$

with relations $x^r = 0$. The Nakayama algebras $N(n, 2)$ are easily proven to be of cyclotomic type, while those of the form $N(n, 3)$ are of cyclotomic type as consequence of lengthly considerations in [7]. The case $r = 4$ is more representative: $N(n, 4)$ is of cyclotomic type for all $0 \leq n \leq 100$ except for $n = 10, 22, 30, 42, 50, 62, 70, 82$ and 90. This claim is obtained as a consequence of methods developed in the paper.

We are interested in results which use only the knowledge of traces of powers of the Coxeter matrix. There are reasons that make such criteria desirable:

(a) Let A be a triangular algebra. From [3] we know that

$$-\mathrm{Tr}\,(\phi_A) = \sum_{k=0}^{\infty} (-1)^k \dim_K H^k(A)$$

where $H^k(A)$ are the spaces associated to the k-th Hochschild cohomology. We say that A is *accessible* from B, if there exists a chain of algebras $A_1 = B, A_2, \ldots, A_s = A$ such that A_{i+1} is a one-point extension or coextension of A_i by an indecomposable A_i-module N_i; if moreover, the N_i are exceptional, that is, the extension rings $\mathrm{Ext}^*_A(N_i, N_i)$ are trivial, then we say that A is *strongly accessible* from B. In the latter case, $H^*(A) \cong H^*(B)$ as rings. If B is one-dimensional, we simply say that A is accessible (resp. strongly accessible). In particular, strongly accessible algebras have $\mathrm{Tr}\,(\phi_A) = -1$. The converse also holds:

Theorem 1. *A finite dimensional accessible algebra A then it is strongly accessible if and only if* $\mathrm{Tr}\,(\phi_A) = -1$.

(b) Clearly, if A is of cyclotomic type then $|\mathrm{Tr}\,(\phi_A)^k| \leq n$, for $k \geq 0$. In Section 3 we show that the converse holds.