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The set of unattainable points for the Rational Hermite Interpolation Problem ☆

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We describe geometrically and algebraically the set of unattainable points for the Rational Hermite Interpolation Problem (i.e. those points where the problem does not have a solution). We show that this set is a union of equidimensional complete intersection varieties of odd codimension, the number of them being equal to the minimum between the degrees of the numerator and denominator of the problem. Each of these equidimensional varieties can be further decomposed as a union of as many rational (irreducible) varieties as input data points. We exhibit algorithms and equations defining all these objects.

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1. Introduction

Let $\mathbb{K}$ be a field, k, l, $n_1, \ldots, n_l \in \mathbb{Z}_{>0}$ with $k \leq n := n_1 + \ldots + n_l$. For $u_1, \ldots, u_l \in \mathbb{K}$ with $u_i \neq u_j$ if $i \neq j$, and $v_{i,j} \in \mathbb{K}$ with $i = 1, \ldots, l$, $j = 0, \ldots, n_i - 1$, the Rational Hermite Interpolation Problem (RHIP) associated with this data as stated in [11,8,12,14], is the following: decide if there exist – and if so compute – polynomials $A(x)$, $B(x) \in \mathbb{K}[x]$ of degrees bounded by $k-1$ and $n-k$ respectively such that $B(u_i) \neq 0$ for all $i = 1, \ldots, l$, and

$$\left(\frac{A}{B}\right)^{(j)}(u_i) = j!v_{i,j},\ i = 1, \ldots, l,\ j = 0, \ldots, n_i - 1. \tag{1}$$

The factorial in the equation above is introduced to simplify some of the formulas that will appear later. For (1) to be as general as possible, we need to impose $\mathrm{Char}(\mathbb{K}) = 0$ or $\mathrm{Char}(\mathbb{K}) \geq \max\{n_1, \ldots, n_l\}$. When $l = n$ (i.e. $n_1 = \ldots = n_l = 1$), the RHIP coincides with the classical Rational Interpolation Problem [3,17,13]. If $k = n$, the RHIP descends to the well-known Hermite Interpolation Problem. But in contrast with it, there is not always a solution for the RHIP for any input data. For instance, if we set $k = 2$, $l = 2$, $n_1 = 2$, $n_2 = 1$, $u_1 = 1$, $u_2 = 2$, $v_{1,0} = 1$, $v_{1,1} = 0$, $v_{2,0} = 0$, one can check straightforwardly that there is no solution for (1), see Example 1.2 below.

The standard approach to this problem from both an algorithmic and complexity point of view is via the Extended Euclidean Algorithm as it is explained in [16] (see also [1], and §2.1 in this text). There are also alternative approaches by using structured matrices ([15,2]), barycentric coordinates ([13,14]), orthogonal polynomials ([6,7]), and computation of syzygies ([10]). Barycentric coordinates seem to be stable when working with approximate data, but not very fast, while the use of orthogonal polynomials is efficient thanks to the use of Jacobi's method for inverting matrices, but their results are limited to the rational interpolation problem (without multiplicities) only. Parametric representations of the solutions in general situations can be found in [9,5].

In all the previous results, optimal bounds of complexity are achieved, and parametric expressions for $A(x)$ and $B(x)$ are given when they exist, but an explicit description of the set of the so called unattainable points for the RHIP, i.e. the set of data $\{u_i, v_{i,j}\}$ such that the RHIP does not have solutions, cannot be obtained straightforwardly from these approaches. The purpose of this paper is to characterize them both geometrically and algebraically. Our main result, given in Theorems 1.1 and 1.4 states that the set of ill-posed point is a union of $\min\{k-1, n-k\}$ equidimensional complete intersection varieties of odd codimension. Moreover, each of these varieties can be further decomposed as a union of l rational (irreducible) varieties. As a by-product, we will produce explicit expressions for the solution for this problem valid in different regions of the space of parameters, and alternative algorithms based only in elementary Linear Algebra, without the need of applying neither barycentric coordinates, nor the Euclidean Division Algorithm. The complexity of solving these problems as well as the extension of the methods in [6] to the RHIP will be the subject of future work.